VOL. 23 No. 270

WIRELESS ENGINEER

MARCH 1946

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EDITORIAL

Aerial Resistance and Cable Impedance

A READER has propounded the following problem for our consideration. "It is known that, for a coaxial transmission line in which the losses are wholly concentrated in the conductors, there exists an optimum ratio of outer to inner conductor diameter at which the attenuation loss of the line is a minimum. It so happens that this ratio provides at high frequencies a characteristic impedance of 72 ohms. It is also known that the radiation resistance of a half-wave dipole in free space is 72 ohms. Is this interesting relation a mere coincidence or is it likely that there exists a real physical identity which is demonstrable?"

The first question to decide is whether the facts are as stated. Let us consider the coaxial line and determine the characteristic impedance when made with the optimum ratio of radii.

The Coaxial Line

In a coaxial line the inductance and capacitance per unit length are given by the formulae.*

$$L = 2 \log r_2/r_1 \times 10^{-9} \text{ henry}$$

$$C = \frac{1}{2 \log r_2/r_1} \times \frac{10^{-11}}{9} \text{ farad}$$

At high frequencies the currents will be confined to a thin surface layer the depth of which will depend only on the material and frequency. The resistance per unit length will therefore be inversely proportional to the periphery and we may write $R = \rho/r_1 + \rho/r_2$ where ρ is not the specific resistance but a constant for a given material and frequency.

For the propagation constant we can write $P = \alpha + j\beta$ where, since we assume negligible leakance and a small value of $R/\omega L$, we can put

$$\alpha = \frac{R}{2}\sqrt{\frac{C}{L}} = \frac{\rho}{2} \cdot \frac{r_1 + r_2}{r_1 r_2} \cdot \frac{1}{2 \log r_2/r_1} \cdot \frac{1}{30}$$

$$= \frac{\rho}{120}\left(\frac{1}{r_1} + \frac{1}{r_2}\right) \frac{1}{\log r_2/r_1}$$

If now we regard the radius r_2 of the outer conductor as fixed and wish to determine the value of r_1 to give a minimum value of the attenuation constant α we must differentiate this expression with respect to r_1 and equate to zero.

$$\frac{d}{dr_1}\left\{\frac{r_1 + r_2}{r_1 r_2} \cdot \frac{1}{\log r_2/r_1}\right\}$$

$$= \frac{1}{r_1^2 \log r_2/r_1}\left(\frac{r_1 + r_2}{r_2} \cdot \frac{1}{\log r_2/r_1} - 1\right)$$

For this to be zero

$$\log r_2/r_1 = 1 + r_1/r_2$$

and by plotting a few values of $\log r_2/r_1$ it is easily seen that this condition holds when $r_2/r_1 = 3.597$ and $\log r_2/r_1 = 1.278$.

If r_2/r_1 be given this optimum value the characteristic impedance Z_0 will be given by the formula

$$Z_0 = \sqrt{(L/C)} = 60 \log r_2/r_1$$

$$= 60 \times 1.278 = 76.7 \text{ ohms.}$$

in which we have neglected the effect of the resistance; if this were taken into account

* All the logarithms in this article are to the base e.

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and by plotting a few values of $\log r_2/r_1$ it is easily seen that this condition holds when $r_2/r_1 = 3.597$ and $\log r_2/r_1 = 1.278$.

If r_2/r_1 be given this optimum value the characteristic impedance Z_0 will be given by the formula

$$Z_0 = \sqrt{(L/C)} = 60 \log r_2/r_1$$

$$= 60 \times 1.278 = 76.7 \text{ ohms.}$$

in which we have neglected the effect of the resistance; if this were taken into account

* All the logarithms in this article are to the base e.

it would slightly increase Z_0. It is interesting to note that this characteristic impedance is independent of the size of the line so long as it has the correct ratio of outer and inner radii, but the above expression for the attenuation constant α shows that it decreases as the size is increased. Our correspondent gave no indication of the source from which he obtained the value of 72 ohms for the impedance.

The Half-wave Dipole.

The radiation resistance of a half-wave dipole is a much more difficult matter. In the Editorial of April, 1945, we showed how it could be calculated from fundamental principles on making certain simplifying assumptions, *viz.*, that the length of the dipole is exactly half a wave-length and that the current distribution is sinusoidal. On these assumptions we showed that the radiation resistance was 73.2 ohms. This immediately suggests that perhaps, if other assumptions were made, approximating more closely to the actual facts, the value might approach more closely to the 76.7 ohms found for the coaxial line. This is not so, however. In *Hochfrequenztechnik und Elektroakustic* of 1934, p. 166, Siegel and Labus calculate the value 73.3 ohms for the radiation resistance when $\lambda = 2l$, but show that to obtain resonance l/λ must be decreased below 0.5 by an amount depending on the size of the conductor, and that this causes a rapid decrease in the radiation resistance, thus removing it still further from the impedance of the coaxial line.

If there were any physical relationship between the radiation resistance of a dipole and the characteristic impedance of a transmission line, one would surely expect to find it in the case of two parallel wires rather than in a coaxial line.

A Line of Two Parallel Wires

In a line consisting of two parallel wires of radius r at a distance D between centres, the inductance, capacitance and resistance per cm of length at high frequencies are given by the formulae :—

$$L = 4 \log \left(\frac{D}{2r} + \sqrt{\left(\frac{D}{2r}\right)^2 - 1} \right) \times 10^{-9} \text{ henry}$$

$$C = \frac{1}{4 \log \left(\frac{D}{2r} + \sqrt{\left(\frac{D}{2r}\right)^2 - 1} \right)} \times \frac{10^{-11}}{9} \text{ farad}$$

$$R = 0.825 \sqrt{f} \, \frac{(D/2r) \times (1/r)}{\sqrt{\left(\frac{D}{2r}\right)^2 - 1}} \, 10^{-9} \text{ ohm}$$

This last formula assumes the wires to be of copper. The complexity of the formulae is due to the so-called proximity effect, that is, to the non-uniform distribution of charge and current on the surface of the wires. As before, we have for the attenuation constant $\alpha = \frac{1}{2} R\sqrt{(C/L)}$ which now becomes

$$\alpha = \frac{0.34\sqrt{f}}{10^{11}} \cdot \frac{D/2r}{r\sqrt{\{(D/2r)^2 - 1\}}} \cdot \frac{1}{\log \left[D/2r + \sqrt{\{(D/2r)^2 - 1\}} \right]}$$

We now assume that the frequency and the distance between the wires are fixed, and determine the value of the radius r that makes α a minimum by equating $d\alpha/dr$ to zero. This leads to the condition that

$$\frac{D}{2r} \cdot \frac{1}{\sqrt{\left(\frac{D}{2r}\right)^2 - 1}} \cdot \frac{1}{\log \left(\frac{D}{2r} + \sqrt{\left(\frac{D}{2r}\right)^2 - 1} \right)} + \frac{\left(\frac{D}{2r}\right)^2}{\left(\frac{D}{2r}\right)^2 - 1} = 2$$

If $\frac{D}{2r}$ be put successively equal to 2.2, 2.3, and 2.4 the left-hand side is found to equal 2.05, 1.985 and 1.9335, and on plotting these we find that to give exactly 2.0, $D/2r$ must be 2.28. This then is the optimum relation between D and $2r$. With this value of $D/2r$

$$\log \left(\frac{D}{2r} + \sqrt{\left(\frac{D}{2r}\right)^2 - 1} \right) = \log (2.28 + 2.05)$$
$$= \log 4.33 = 1.461$$

For the characteristic impedance we have (neglecting the effect of resistance)

$$Z_0 = \sqrt{\frac{L}{C}} = 120 \log \left(\frac{D}{2r} + \sqrt{\left(\frac{D}{2r}\right)^2 - 1} \right)$$
$$= 120 \times 1.461 = 175.5 \text{ ohms.}$$

This is 2.4 times the radiation resistance of the half-wave dipole. We are forced to the conclusion that the interesting relationship is non-existent.

G. W. O. H.

CARRIER-FREQUENCY AMPLIFIERS*

Transient Response with De-tuned Carrier

By C. C. Eaglesfield

(*Mullard Radio Valve Company*)

SUMMARY.—A treatment is given of the problem of the transient response of amplifiers when the carrier frequency may differ from the central frequency of the amplifier. The importance of the depth of modulation of the input waveform is investigated : it is shown that there are good reasons for taking the modulation depth small. Several curves are given, showing typical responses.

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1. Introduction
List of Main Symbols
2. Step Response (Full Modulation)
3. Steady State Characteristic
4. Step Response (Small Modulation)
5. Numerical Examples

1. Introduction

IN a previous article[1] the writer has given the solution of the envelope response of a multistage carrier-frequency amplifier, with certain limitations on the analysis. These limitations were that successive valves in the chain should be coupled by staggered single circuits, that is circuits tuned to adjacent frequencies, or by double circuits ; and that the carrier frequency should be the same as the central frequency of the amplitude-frequency characteristic of the amplifier.

This second condition is now removed, and the solution is given for the case where the carrier frequency differs from the central frequency of the amplifier.

The test input waveform used before was the carrier frequency modulated by a unit step, as shown in Fig. 1 (*a*). It soon became apparent, with the unrestricted input frequency, that this was a bad choice. The test input waveform now used is as shown by Fig. 1 (*c*), where the modulation is still a unit step, but the depth of modulation is small instead of, as formerly, 100 per cent. By restricting the depth of modulation a great simplification is made in the analysis, and it is shown that there exists a " modulation impedance " connecting the output current envelope with the input modulation voltage, this impedance being linear when the depth of modulation is small. In this way the relation between the current envelope and the modulation voltage is made exactly analogous to the more simple case of the current in an impedance subjected to a voltage.

It is worth remarking that if a receiver is tested experimentally, for instance, by using a signal generator modulated by square waves, the depth of modulation should be small. If a large modulation depth is used, the output waveform will not be symmetrical about the time axis.

It so happens that for the case considered previously, the envelope response is the same whether the input is according to Fig. 1 (*a*) or 1 (*c*) ; so that no modification need be made in the results already given.

Since a further parameter is now introduced—the relation between the carrier frequency and the amplifier central frequency—it is hardly practicable to give comprehensive numerical results ; another difficulty is that the response curves are now more complicated, and cannot be summarized by the two descriptive figures, speed and overshoot, previously used by the writer.

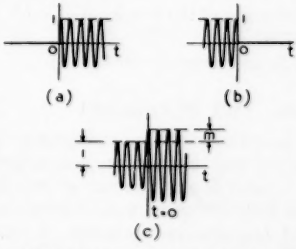

Fig. 1. *The three input waveforms used in the analysis. " a " and " b " full modulation, " c " small modulation.*

Since the general form of the expressions for double circuits is the same as for an equal number of staggered pairs, only the latter case has been considered, as the analysis is rather simpler.

As before, the analysis is simplified by making the carrier frequency, together with

* MS. accepted by the Editor, July 1945.

the central frequency of the amplifier, tend to infinity; the difference-frequency being kept constant.

The treatment given follows the sequence of the writer's own approach; thus in Section 2 the envelope is calculated for N staggered-pairs driven by the waveform of Fig. 1 (a), and an explicit but cumbersome result is obtained; from this the steady-state equations are obtained in Section 3. In Section 4 the difficulties due to using 100 per cent. modulation are discussed, and general small-modulation equations are developed, which lead to a simpler result. Some numerical results are given in Section 5, which show the modulation response for four staggered-pairs with different degrees of adjustment of stagger and carrier frequency.

List of Main Symbols

n The number of valves in the chain.
N Equal to $n/2$; an integer.
p The differential operator d/dt (with reservations).
ω_0 The angular frequency of the carrier.
$(\omega_0 + \omega_1)$ The angular frequency of resonance of one set of circuits of the chain.
$(\omega_0 + \omega_2)$ The angular frequency of resonance of the other set of circuits of the chain.
g The mutual conductance of each valve.
C The total capacitance per valve stage.
R The parallel resistance across each circuit.
a The ratio $1/2CR$.
k The "stagger coefficient"; equal to $\frac{\omega_1 - \omega_2}{2a}$
h The "detune coefficient"; equal to $\frac{\omega_1 + \omega_2}{2a}$
θ The steady-state phase change.
z Equal to kat.
x Equal to at.
$J_\nu(z)$ The Bessel function of the first kind of order ν and argument z.
$\mathbf{1}$ The Heaviside unit step function.

2. Step Response (Full Modulation)

Consider an amplifier chain consisting of n stages, of which Fig. 2 is a typical stage. For each stage C and R are the same, but in alternate stages L has one of two values, so that one set of circuits is resonant at one frequency and the other set, equal in number, at a second frequency. n is an even integer.

A voltage is injected into the grid of the first valve: the form of this voltage is a third frequency modulated by a unit step, as shown in Fig. 1 (a). The voltage at the end of the amplifier will be calculated, and then its envelope.

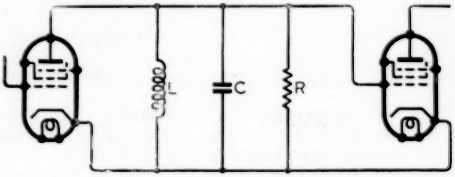

Fig. 2. *A typical amplifier stage.*

As a means of simplifying the various expressions that will occur, the carrier frequency is made to tend to infinity, while the difference frequencies between the carrier frequency and the resonant frequencies of the circuits are kept constant.

The stage gain is gZ, where g is the mutual conductance of each valve, and Z is the impedance of the circuit. This assumes that the anode A.C. resistance of each valve is very much larger than the circuit impedance.

The admittance of one of the circuits is:—

$$\frac{1}{Z} = \frac{1}{R} + pC + \frac{1}{pL}$$

or $$Z = \frac{1}{C}\,\frac{p}{(p+a)^2 + \omega^2}$$

where $a = \frac{1}{2RC}$ and $\omega^2 = \frac{1}{LC} - a^2$

We can thus write for the impedances of two successive circuits:—

$$Z = \frac{1}{C}\,\frac{p}{(p+a)^2 + (\omega_0 + \omega_1)^2}$$

and

$$Z' = \frac{1}{C}\,\frac{p}{(p+a)^2 + (\omega_0 + \omega_2)^2}$$

where ω_0 will be taken as the carrier angular frequency, and ω_1, ω_2 as the difference angular frequencies.

The gain for two stages is g^2ZZ'.

With the Heaviside notation the input voltage is $\cos \omega_0 t\ \mathbf{1}$; that is, the real part of $e^{j\omega_0 t}\mathbf{1}$.

Thus the voltage at the anode of the second valve is

$$\left(\frac{g}{C}\right)^2 \frac{p^2}{[(p+a)^2 + (\omega_0 + \omega_1)^2]\,[(p+a)^2 + (\omega_0 + \omega_2)]^2}\, e^{j\omega_0 t}\,\mathbf{1}$$

the real part to be taken.

By Heaviside's Shifting Theorem, the exponential term $e^{j\omega_0 t}$ can be shifted outside the operator by writing $(p + j\omega_0)$ in the place of p :—

$$e^{j\omega_0 t}\left(\frac{g}{C}\right)^2 \frac{(p+j\omega_0)^2}{[(p+a+j\omega_0)^2+(\omega_0+\omega_1)^2]\,[(p+a+j\omega_0)^2+(\omega_0+\omega_2)^2]}\,1$$

Now factorise the denominator :—

$$e^{j\omega_0 t}\left(\frac{g}{C}\right)^2 \frac{(p+j\omega_0)^2}{[p+a+j\,(2\omega_0+\omega_1)]\,[p+a-j\omega_1]\,[p+a+j\,(2\omega_0+\omega_2)]\,[p+a-j\omega_2]}\,1$$

Now let $\omega_0 \to \infty$

then $$\frac{(p+j\omega_0)^2}{[p+a+j\,(2\omega_0+\omega_1)]\,[p+a+j\,(2\omega_0+\omega_2)]} \to \tfrac{1}{4}$$

and our expression becomes :—

$$e^{j\omega_0 t}\left(\frac{g}{2C}\right)^2 \frac{1}{(p+a-j\omega_1)\,(p+a-j\omega_2)}\,1$$

Having simplified the operator by the removal of ω_0, it is now advisable to start moulding it into a standard form.

Note that

$$\begin{aligned}(p+a-j\omega_1)\,(p+a-j\omega_2) &= \left(p+a-j\frac{\omega_1+\omega_2}{2}\right)^2+\left(\frac{\omega_1-\omega_2}{2}\right)^2\\ &= (p+a-jb)^2+\omega^2\end{aligned}$$

where $b = \dfrac{\omega_1+\omega_2}{2}$ and $\omega = \dfrac{\omega_1-\omega_2}{2}$

The expression for the voltage is now :—

$$e^{j\omega_0 t}\left(\frac{g}{2C}\right)^2 \frac{1}{(p+a-jb)^2+\omega^2}\,1 \quad .. \quad (1)$$

This can now be simplified by the removal of an exponential term :—

$$e^{j\omega_0 t}\left(\frac{g}{2C}\right)^2 \int_0^t e^{-(a-jb)t}\,\frac{p}{p^2+\omega^2}\,1\,dt$$

This is the voltage after two valves ; after n valves the expression is :—

$$e^{j\omega_0 t}\left(\frac{g}{2C}\right)^n \int_0^t e^{-(a-jb)t}\,\frac{p}{(p^2+\omega^2)^{n/2}}\,1\,dt \quad .. \quad .. \quad (2)$$

The operator $\dfrac{p}{(p^2+\omega^2)^{n/2}}\cdot 1$ is a standard form.

See for example, McLachlan[2] Appendix II, equation 35, from which is obtained

$$\frac{p}{(p^2+\omega^2)^{n/2}}\,1 = \frac{\sqrt{\pi}}{(n/2-1)\,!}\left(\frac{t}{2\omega}\right)^{\frac{n-1}{2}} J_{\frac{n-1}{2}}(\omega t) \quad .. \quad .. \quad (3)$$

Thus, substituting from equation (3) in equation (2) :—

$$e^{j\omega_0 t}\left(\frac{g}{2C}\right)^n \frac{\sqrt{\pi}}{(n/2-1)\,!}\left(\frac{1}{2\omega}\right)^{\frac{n-1}{2}} \left[\begin{array}{l}\displaystyle\int_0^t e^{-at}(\cos bt)\,t^{\frac{n-1}{2}} J_{\frac{n-1}{2}}(\omega t)\,dt \\ \displaystyle\quad + j\int_0^t e^{-at}(\sin bt)\,t^{\frac{n-1}{2}} J_{\frac{n-1}{2}}(\omega t)\,dt\end{array}\right] \quad (4)$$

Before finding the envelope of equation (4), rewrite it after making several new substitutions :—

Put
$$\left.\begin{aligned}\omega t &= z\\ n/2 &= N\\ b/a &= h\\ \omega/a &= k\\ \frac{\sqrt{2\pi N}\,N^N e^{-N}}{N\,!} &= H(N)\end{aligned}\right\} \quad .. \quad .. \quad (4a)$$

Equation (4) then becomes :—

$$e^{j\omega_0 t}\left(\frac{g}{2Ca}\right)^n \frac{1}{2^N}\frac{1}{k^n} H(N)\,[I_1(z) + jI_2(z)] \quad .. \quad .. \quad (5)$$

where

$$\left.\begin{matrix}I_1\\ I_2\end{matrix}\right.(z) = \int_0^z e^{N-z/k}\left.\begin{matrix}\cos\\ \sin\end{matrix}\right.\left(\frac{h}{k}z\right)\left(\frac{z}{N}\right)^{N-\frac{1}{2}} J_{N-\frac{1}{2}}(z)\,dz \quad .. \quad .. \quad (6)$$

The real part of equation (5) is the voltage from the *nth* valve. The envelope is given by the modulus, since ω_0 is tending to infinity. The envelope after n valves is thus

$$V_n = \left(\frac{g}{2Ca}\right)^n \frac{1}{2^N}\frac{1}{k^n} H(N)\,[I_1^2(z) + I_2^2(z)]^{\frac{1}{2}} \quad .. \quad .. \quad (7)$$

At this stage it is reasonable to pause for a moment. We now have, in equation (7), an expression for the envelope which is explicit, but not very pleasant. It contains two parameters, h and k. The parameter k is a measure of the separation between the resonant frequencies of the circuits; and h is a measure of the separation between the carrier frequency and the circuit frequencies. k might be called the "stagger" coefficient and h the "detune" coefficient. The function $H(N)$ tends to unity for large N. The Bessel function is of half integral order (since n is an even integer); it is tabulated to some extent, and can be expressed in terms of a finite number of elementary functions.

For numerical values of the parameters, the envelope can be calculated numerically.

3. Steady State Characteristic

Now consider the case when the input voltage is in the form of an unmodulated frequency, merely cos $\omega_0 t$. This corresponds to the case of Section 2 when t is very large, and the output voltage is obtained in the usual way by putting $p = 0$.

Equation (1) gives the voltage after two stages; after n stages it is:—

$$e^{j\omega_0 t}\left(\frac{g}{2C}\right)^n \frac{1}{[(p + a - jb)^2 + \omega^2]^{n/2}}\,\mathbf{1} \quad .. \quad .. \quad (8)$$

Put $p = 0$, and remove the step sign:—

$$e^{j\omega_0 t}\left(\frac{g}{2C}\right)^n \frac{1}{[(a - jb)^2 + \omega^2]^{n/2}}$$

$$= e^{j\omega_0 t}\left(\frac{g}{2Ca}\right)^n \frac{1}{[(1 - jh)^2 + k^2]^{n/2}}$$

$$= e^{j\omega_0 t}\left(\frac{g}{2Ca}\right)^n \frac{1}{[(1 + k^2 - h^2)^2 + 4h^2]^{n/4}}\, e^{\frac{jn\theta}{2}} \quad .. \quad .. \quad (9)$$

where $\tan\theta = \dfrac{2h}{1 + k^2 - h^2}$ .. .. (10)

The amplitude of expression (9) is:—

$$\left(\frac{g}{2Ca}\right)^n \frac{1}{[(1 + k^2 - h^2)^2 + 4h^2]^{n/4}} \quad .. \quad (11)$$

The amplitude-frequency characteristic in the steady state condition is given by equation (11): h being regarded as a measure of the "detune" of the carrier frequency. When h is zero the carrier frequency is symmetrically in the centre of the amplitude—frequency characteristic.

Several things of interest can be got from equation (11). The first is the "detune ratio," defined as the amplitude for a certain value of h divided by the amplitude at the centre of the characteristic ($h = 0$). This can be written down from equation (11):—

$$\text{"Detune ratio"} = \left[\frac{(1 + k^2)^2}{(1 + k^2 - h^2)^2 + 4h^2}\right]^{n/4} \quad .. \quad .. \quad (12)$$

The next thing of interest is the peak to centre ratio of the characteristic. This is obtained by differentiation of equation (11) with respect to h to find the maximum value of the characteristic, and then division by the value at the centre of the characteristic. The details, which are straightforward, are omitted. The result is

$$\text{"Peak to centre ratio"} = \left[\frac{1 + k^2}{2k}\right]^{n/2} \quad (13)$$

provided $k \geqslant 1$. For $k < 1$ the characteristic has only a single peak. $k = 1$ is a critical case for the characteristic, at which it changes from single to double peaks.

The next thing is to observe how to arrange matters so that the amplitude (*i.e.*, the gain of the amplifier chain) is constant for any values of h and k. This requires that

$$\left(\frac{g}{2Ca}\right)^n \frac{1}{[(1 + k^2 - h^2)^2 + 4h^2]^{n/4}}$$

be independent of h and k. This can be done by making a depend on h and k. Suppose $a = a_0$ for $h = k = 0$.

$$\text{Then } a = \frac{a_0}{[(1 + k^2 - h^2)^2 + 4h^2]^{\frac{1}{4}}} \quad .. \quad (14)$$

a_0 corresponds to the case of the circuits all resonant to the same frequency and driven at that frequency. Equation (14) expresses the adjustment of R(Fig. 2) required to keep the gain constant for different amounts of stagger and detune.

4. Step Response (Small Modulation)

We have obtained in Section 2 equation (7) an expression for the step response with full modulation, that is the input has the shape of Fig. 1 (*a*). Now consider the case where

the carrier, instead of being suddenly applied, is suddenly removed. That is, the input has the shape of Fig. 1 (*b*). It might be expected that the new response would be the mirror image of the first response, since that happens when a plain step is applied to or removed from an ordinary linear network.

The waveform of Fig. 1 (*b*) is expressed operationally by

$$\cos \omega_0 t\,(1 - \mathbf{1})$$

and by combination of the equations in Sections 2 and 3 the envelope response can be obtained.

Fig. 3 shows the envelope for a particular case ($n = 8$, $h = 1$, $k = 1$), both for the application and removal of a step. It can be seen that the one is by no means the mirror image of the other, in fact, the shapes are quite different.

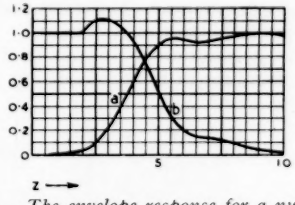

Fig. 3. *The envelope response for a numerical case, $n = 8$, $h = 1$, $k = 1$. "a" is the response to an input of the shape of Fig. 1 (a); "b" is the response to Fig. 1 (b).*

Since there is no obvious reason to favour either Fig. 1 (*a*) or 1 (*b*) in preference to the other, it seems that neither is a convenient test waveform.

A more convenient test waveform is shown in Fig. 1 (*c*), where the amplitude changes suddenly from 1 to $(1 + m)$ at $t = 0$. m is effectively the modulation depth, and the cases considered so far have been equivalent to 100 per cent. modulation.

It will be shown that if m is kept small, the shape of the response is independent of m.

Before dealing with the particular problem, consider the general case of an admittance subjected to a voltage of the form of Fig. 1 (*c*).

The voltage is represented by

$$e^{j\omega_0 t}\,(1 + m\mathbf{1}).$$

Write $A\,(p)$ for the admittance. Then the resulting current is

$$A\,(p)\,e^{j\omega_0 t}\,(1 + m\mathbf{1})$$

Shift the exponential term outside the operator :—

$$e^{j\omega_0 t}\,A(p + j\omega_0)\,(1 + m\,\mathbf{1})$$

This is made up of two terms, of which one is a steady state term

$$A(p + j\omega_0)1 = A(j\omega_0)$$

We can thus rewrite it

$$e^{j\omega_0 t}\,[A(j\omega_0) + mA\,(p + j\omega_0)\,\mathbf{1}]$$

$$= e^{j\omega_0 t}\,A(j\omega_0)\left[1 + m\,\frac{A(p + j\omega_0)}{A\,(j\omega_0)}\,\mathbf{1}\right]$$

To obtain the envelope we take the modulus.

Write M_0 for the modulus of $A(j\omega_0)$

and $B(p)$ for the real part of $\dfrac{A(p + j\omega_0)}{A(j\omega_0)}$

For m very small the modulus is

$$M_0[1 + mB(p)\,\mathbf{1}].$$

Remove from this the steady state term M_0.

There is left $M_0 mB(p)\,\mathbf{1}$.

This is the current resulting from an applied modulation voltage $m\mathbf{1}$.

The expression $M_0 B(p)\,\mathbf{1}$ may be regarded as the "modulation admittance." It has been derived on the assumption that m is vanishingly small, and with that condition it is a linear admittance.

If m is finite the modulation admittance is not linear.

It is convenient to omit the factor M_0, which is merely the steady state gain, and take $B(p)\,\mathbf{1}$ as the "modulation response." Since $B(0) = 1$, this response will become unity for large values of t. If $B(p)\,\mathbf{1} = f(t)$, then $f(t)$ uniquely determines the modulation admittance. The response when any (small) modulation voltage is substituted for the step is obtained from $f(t)$ in the usual way, by Borel's Theorem.

Now apply these results to the particular problem. $A(p + j\omega_0)$ is given by equation (8) :—

$$\left(\frac{g}{2C}\right)^n \frac{1}{[(p + a - jb)^2 + \omega^2]^{n/2}}$$

Thus

$$\frac{A(p + j\omega_0)}{A(j\omega_0)}\,\mathbf{1} = \left[\frac{(a - jb)^2 + \omega^2}{(p + a - jb)^2 + \omega^2}\right]^{n/2}\mathbf{1}$$

and of this the real part is the modulation response.

The denominator is rationalized by the

removal of an exponential term $e^{-(a-jb)t}$ as before, and the numerator in the same way as led to equation (9). Omitting the details of the work, the modulation response is given by

$$A\int_0^z e^{N-z/k}\left\{\cos\left(\frac{h}{k}z - N\theta\right)\right\}\left(\frac{z}{N}\right)^{N-\frac{1}{2}} J_{N-\frac{1}{2}}(z)\,dz \quad .. \quad .. \quad (15)$$

where $$A = \left[\frac{(1+k^2-h^2)^2+4h^2}{4k^4}\right]^{n/4} H(N) \quad .. \quad .. \quad (16)$$

$H(N)$ is given by equation 4(*a*).

θ is given by equation (10).

$z = kat$, $N = n/2$.

The expression (15) tends to unity for large z. It will be observed that (15) is much simpler than (7), so that the use of very small modulation has led to a simpler result in addition to its other advantages. Three degenerations of equation (15) are worthy of remark.

First, $h = 0$.

Then $\theta = 0$, and the expression becomes

$$A\int_0^z e^{N-z/k}\left(\frac{z}{N}\right)^{N-\frac{1}{2}} J_{N-\frac{1}{2}}(z)\,dz \quad .. \quad (17)$$

Second, $k = 0$.

This makes $z = 0$; define a new variable $x = at$.

After some manipulation, equation (15) becomes :—

$$\frac{(1+h^2)^{n/2}}{(n-1)\,!}\int_0^x e^{-x}\,x^{n-1}\cos(hx - N\theta)\,dx \quad .. \quad .. \quad (18)$$

Third, $h = 0$ and $k = 0$.

Equation (18) reduces to :—

$$\frac{1}{(n-1)\,!}\int_0^x e^{-x}\,x^{n-1}\,dx \quad .. \quad .. \quad (19)$$

The expression (19) will be recognised as the incomplete gamma function, which is well tabulated[3].

Inspecting the expressions (16) to (19), it is seen that " stagger " is characterised by a Bessel function in the integrand and " detune " by a sinusoidal function.

5. Numerical Examples

A few examples have been calculated. It is difficult to produce enough evidence on which to base firm conclusions, not because the calculation for any particular case is very laborious, but because of the number of parameters — n, h, k. Accordingly only a single value of n has been taken, $n = 8$, which is high enough to be interesting, but not so high as to be irrelevant ; and a few values of h and k.

The general expression is given by (15), and the first thing to be observed is that it contains a Bessel function of half-integral order. These functions are tabulated[4], but only apparently for integral values of the argument. This is rather wide and the integration needs care.

The numerical integration of a function is likely to daunt the amateur computer, but it is not in fact difficult. Many formulae are generally known for evaluating a single definite integral ; the writer is indebted to Mr. D. P. Dalzell for drawing his attention to the following method of performing a functional integration. The derivation given should not be taken too seriously.

Suppose a function y of x is known at a number of values of x, separated by uniform increments of x. E.g. :—

x	0	1	2	—
y	y_0	y_1	y_2	—

Required $\int_0^1 y\,dx$, $\int_0^2 y\,dx$, etc.

By Taylor's Theorem in finite differences

$$y = (1+\Delta)^x y_0$$

Thus $$\int y\,dx = \frac{(1+\Delta)^x}{\log(1+\Delta)} y_0$$

and

$$\int_0^1 y\,dx = \left[\frac{1+\Delta}{\log(1+\Delta)} - \frac{1}{\log(1+\Delta)}\right] y_0$$

$$= \frac{\Delta}{\log(1+\Delta)}\, y_0$$

$$= [1 + \tfrac{1}{2}\Delta - \tfrac{1}{12}\Delta^2 + \tfrac{1}{24}\Delta^3 - \tfrac{19}{720}\Delta^4 \ldots]\, y_0 \quad .. \quad (20)$$

The rule is therefore to construct a set of Y's such that

$$Y_n = y_n + \tfrac{1}{2}\Delta y_n - \tfrac{1}{12}\Delta^2 y_n + \tfrac{1}{24}\Delta^3 y_n \ldots \quad .. \quad .. \quad (21)$$

and the successive values of the integral are

$$0,\ Y_0,\ Y_0 + Y_1,\ Y_0 + Y_1 + Y_2, \text{ etc.}$$

for $x = 0,\ 1,\ 2,\ 3,$ etc.

Equation (21) is convenient only if a machine for forming differences is available.

If only a normal calculating machine is available, it is better to rewrite (21) in terms of y_n, y_{n+1}, etc. The formula will depend on the order of difference retained. If differences above the third are neglected, then since

$$\Delta y_n = y_{n+1} - y_n$$
$$\Delta^2 y_n = y_{n+2} - 2y_{n+1} + y_n$$
$$\Delta^3 y_n = y_{n+3} - 3y_{n+2} + 3y_{n+1} - y_n$$

equation (21) becomes

$$Y_n = \tfrac{9}{24} y_n + \tfrac{19}{24} y_{n+1} - \tfrac{5}{24} y_{n+2} + \tfrac{1}{24} y_{n+3} \quad .. \quad .. \quad (22)$$

The process therefore consists in multiplying the set of y's first by 9/24, then by 19/24, etc. ; and adding appropriately. It is worth while to write the y's on a separate slip of paper, which can be moved relative to the paper on which the products are recorded : the four products making up the Y's can then be kept in a straight line across the paper.

In practice the process is surprisingly quick and simple.

Modulation response curves for several adjustments of h and k are given in Fig. 4. These have been normalized for equal gain by equation (14), and the time variable is $a_0 t$ where a_0 corresponds to the adjustment of Fig. 4 (*a*). $n = 8$ in all cases. Each curve is accompanied by a diagram of the amplitude-frequency characteristic, not, of course, to scale, the arrow showing the position of the carrier frequency relative to the characteristic. The inclusion of the amplitude characteristic is not to be taken as suggesting more than a cousinly relationship between it and the modulation response ; it is put in chiefly for convenience in assessing the parameters.

The maximum slope and stationary values are inserted on the curves.

While it is dangerous to generalize from such restricted information, a few remarks may be permitted. " Stagger ', alone and " detune " alone are characterized mainly by an overshoot. Detune alone gives a lower slope and more overshoot than stagger alone ; it may well be generally true that for the same amount of overshoot, stagger gives a greater slope than detune. Detune alone gives a preliminary swing in the reverse direction, but of small magnitude compared to the overshoot. A combination of stagger

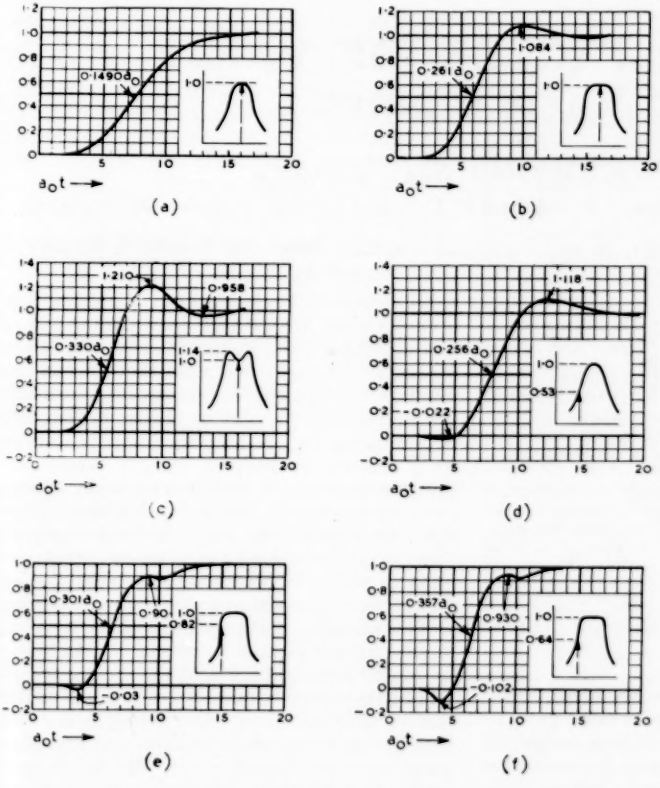

Fig. 4. *Transient response for various values of h and k.*

(*a*) $h = 0$, $k = 0$;
(*b*) $h = 0$, $k = 1$;
(*c*) $h = 0$, $k^2 = 5/3$;
(*d*) $h = \sqrt{2} - 1$, $k = 0$;
(*e*) $h = 0.8$, $k = 1$;
(*f*) $h = 1$, $k = 1$.

and detune makes this reverse swing much more important, the combination being marked by reverse swing and a pause at less than the final value.

Both stagger and detune increase the slope, and both together increase the slope still further. It is difficult to compare the response with stagger and detune with the response for stagger alone, since the shape in the first case is more complicated. It is not easy to summarize the distortion of the response by a single figure, so that while the slopes can be compared the distortions cannot.

Thus, while it seems permissible to increase the slope of the response by stagger alone or detune alone, preferably stagger, it is not clear whether it is advisable to use both at once.

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NOISE FACTOR OF VALVE AMPLIFIERS*

By N. R. Campbell, Sc.D., F.Inst.P., †
V. J. Francis, B.Sc., F.Inst.P., A.M.I.E.E., and E. G. James, Ph.D., B.Sc.

(Communication from the Research Staff of The M.O. Valve Company at the G.E.C. Research Laboratories, Wembley, England)

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1. Introduction

IN this and in later papers, we shall apply the conclusions of two earlier papers[1], [2]‡ to the problems that arise in the design of valve amplifiers. Little has to be added by way of principle to what has already been established; but, since the algebraical development necessary to apply the principles in their most convenient form is unavoidably long and complicated, it will be well to indicate at the start the problems to which the discussion is immediately directed and the nature of the results that are attained.

We suppose, then, that an amplifier has been designed with a pass-band from f_1 to f_2 and a defined cut-off at each end. We want to know whether the noise factor can be improved, subject to some condition limiting the permissible distortion of the pass-band, by prefixing to the amplifier another stage with known properties, *e.g.*, a common-cathode pentode, a common-cathode triode, or a common-grid triode.§ We shall show that, so long as the amplifier and the prefixed stage belong to the class of valve circuits to which B refers, this problem can be solved formally without any approximations or simplifying assumptions, and without neglecting any circuit elements. In practice, however, assumptions may be necessary in all but the simplest cases because some of the factors that enter into the problem are not readily ascertained; further in complex problems the retention of all the circuit elements leads to algebra, and even to

* MS. accepted by the Editor, November 1945.
† Late Member of the Staff of the G.E.C. Research Laboratories.
‡ These papers are referred to throughout as A and B, respectively.
§ The use of the word "common" in this connection denotes that electrode which is common to input and output circuits.

arithmetic, of almost prohibitive complexity. On the other hand, it is perfectly feasible to discuss the effect of some factors that are ignored completely in the conventional treatment of the problem (*e.g.*, the inductances in the earth leads).

In stating the problem the term " noise factor " has been mentioned. None of the previous definitions of this term that we know has the generality required for our purpose ; one of our results will be a better definition. But it should be insisted here that our definition rests on the same basis as all others, namely, the comparison of an " ideal " with a " practical " receiver, when both are fed by the same signal generator. The ideal receiver is supposed to have the same gain at any frequency as the practical receiver, but it generates no noise ; when the ideal receiver is used, all the noise arises in the signal generator. If S/N denotes signal/noise ratio, and the suffix O refers to the ideal receiver and no suffix to the practical receiver, then the noise factor N is defined as $(S/N)o/(S/N)$.

2. General Formulae for Noise

We shall start by re-stating those results of the two previous papers that will be used in this paper.

Noise is due to the happening of events random in time with a mean frequency λ. If these events act on a linear system, each of them, if it acted alone, would produce an effect

$$y = s(t) \qquad .. \qquad .. \qquad .. \qquad (2.1)$$

at the place where the noise is measured, and the effects of different events would be simply additive. We suppose that the events are effectively instantaneous, that is to say, that the duration of each is very small compared with any period characteristic of the system that it affects. This assumption implies that the " transit time " of the electrons in the valve is effectively zero. Many of the propositions about to be stated are actually true even if there is transit time ; but since not all are true, the assumption is made generally. The system is dissipative, so that the effect of each event ultimately dies away, and $s(\infty) = 0$.

In these circumstances the noise, measured by the mean square deviation of y from its mean, is given by

$$\overline{(y - \bar{y})^2} = \lambda \int_0^\infty |s(t)|^2 \, dt = 2\lambda \int_0^\omega |S(j\omega)|^2 \, d\omega \qquad .. \qquad .. \qquad (2.2)$$

where $S(j\omega)$ is the Fourier transform of $s(t)$. Since λ depends only on the nature of the events, the problem of finding the effect of the amplifier on the observed noise reduces to that of finding $s(t)$ or $S(j\omega)$.

This problem is attacked in B. Consideration is there limited to a particular class of single-stage amplifiers (called here class X), which includes many—but perhaps not all—R.F. single-stage amplifiers.* This class is that in which the relation between currents and voltages is linear ; that all the connections between the terminals of the valve(s) are, or are equivalent to, a set of impedors each connecting one pair of terminals and independent of the input impedances of the valve(s) themselves—and that any signal E.M.F. is introduced in series, actually or effectively, with one of these impedors. The use of the expression "equivalent" here allows some, but not all, circuits involving mutual inductance to be included. The terminals of a valve consist of its electrodes together, possibly, with an " earth " terminal which is in general the common terminal of the signal input and output circuits, but is not identical with (*i.e.*, connected by an element of zero impedance to) any of the electrodes.

In amplifiers of class X the events that constitute noise can be simulated by instantaneous changes in the E.M.F. of impedanceless generators, one in series with each of the impedors. Then the transform $S(j\omega)$ is closely related to the " gain " between each simulating generator and the terminals at which the noise is measured ; by the " gain " A_{klpq} between the generator in series with the impedor connecting terminals (p, q) and the output terminals (k, l), we mean the generally complex and frequency-dependent ratio V_{kl}/E_{pq}, where V_{kl} is the voltage appearing between the terminals (k, l) in virtue of the application of a sinusoidal E.M.F. E_{pq} of frequency $\omega/2\pi$ in series with the impedor.

If the noise is thermal noise, arising from a resistor that forms part, or the whole of, the impedor between the terminals (p, q), then

* In paper B, it was stated that circuits in which an admittance is common to two branches each connecting a pair of electrodes were excluded from the theory there given. This, in fact, is incorrect and consideration shows that such circuits can be dealt with in the obvious way suggested by the theory. The only circuits therefore excluded from the theory are the special cases of mutual inductance mentioned in paper B.

(2.2) becomes Johnson's formula

$$\overline{(v-\bar{v})^2}\frac{2k'I}{\pi}\int_0^\infty R_{pq}|A_{kl,\,pq}(j\omega)|^2\,d\omega \quad (2.3)$$

where v is the noise voltage at the terminals (k, l), R_{pq} the real and generally frequency-dependent part of the impedance of the impedor, T its absolute temperature, and k' Boltzmann's constant. (The dash is attached to the usual symbol for this constant in order to distinguish it from the symbol denoting a terminal.)

If the noise is shot noise, arising from a stream of electrons that leave the electrode p and either return to the cathode in virtue of space charge or reach a single electrode q, then

$$\overline{(v-\bar{v})^2}=\frac{I_0\epsilon\Gamma^2}{\pi}\int_0^\infty\frac{1}{\omega^2C^2_{pq}}|A_{kl,\,pq}(j\omega)|^2\,d\omega \quad .. \quad .. \quad (2.4)$$

where I_0 is the current conveyed by the electrons arriving at q, ϵ is the electronic charge, C_{pq} is the capacitance between the terminals (p, q) including the mutual capacitance of the electrodes (p, q), and Γ^2 is the space-charge-reduction factor so denoted by North.[3] If, on the other hand, the electron stream permanently leaving the electrode p is divided between two electrodes q, r in such a ratio that the current α_qI_0 reaches the former and the current α_rI_0 reaches the latter, with $\alpha_q+\alpha_r=1$, then (2.4) is to be replaced by

$$\overline{(v-\bar{v})^2}=\frac{I_0\epsilon\alpha_q}{\pi}(\alpha_q\Gamma^2+\alpha_r)\int_0^\infty\frac{1}{\omega^2C^2_{pq}}|A_{kl,\,pq}(j\omega)|^2\,d\omega$$
$$+\frac{I_0\epsilon\alpha_r(\alpha_r\Gamma^2+\alpha_q)}{\pi}\int_0^\infty\frac{1}{\omega^2C^2_{pr}}|A_{kl,\,pr}(j\omega)|^2\,d\omega \quad .. \quad .. \quad .. \quad .. \quad (2.5)$$
$$-\frac{2I_0\epsilon\alpha_q\alpha_r(1-\Gamma^2)}{\pi}\int_0^\infty\text{real part of}\left\{\frac{1}{j\omega C_{pq}}A_{kl,pq}(j\omega)\right\}\left\{\frac{1}{j\omega C_{pr}}A_{kl,pr}(j\omega)\right\}^*d\omega$$

where the asterisk denotes the complex conjugate of the expression in { } to which it is attached.

If the random events in two different sources of noise are uncorrelated, i.e., if the chance of an event's happening in one source is independent of the occurrence of events in the other source, then, but only then, the value of $\overline{(v-\bar{v})^2}$ for the combination of the two sources is the sum of its values for the individual sources; the sources may then be termed independent. The final confirmation that any two sources are in fact independent must often be decided ultimately by experimental comparison of the noise of the combination with the noises of the components; but the condition for independence just given suggests certain general rules that are found to be valid. Thus the random events in resistors occur in the material of which they are composed; those of a stream of electrons in the cathode from which they proceed. Accordingly we suspect—and find—that thermal noise is always independent of shot noise; that two resistors are independent when the material composing one includes no part of the material composing the other; and that streams of electrons are independent when they proceed from different cathodes, or from distinct parts of the same cathode. The anode and screen currents of a pentode are not independent, because they proceed from the same part of the same cathode.

3. General Formulae for the Gain

The application of formulae (2.3, 4, 5) requires a knowledge of the gains $A_{kl,\,pq}$. It is proved in B that, if the amplifier is of class X, these gains can be expressed in terms of a determinant Δ, whose elements involve the admittances of the impedors, the cofactors of Δ, the admittances themselves, and (if they exist) the mutual inductances between the impedors. Some of the formulae established will now be given and illustrated by the examples of the common-cathode pentode and the common-grid triode, in order that their significance may be clear.

If there is an earth terminal, not identical with any of the electrodes and forming a common terminal of input and output

circuits (" input " here means signal input), then Δ has n^2 elements, where n is the number of the electrodes, which are distinguished by the numerals 1 to n; the earth terminal is denoted by 0. Δ will be written

$$\Delta = \begin{vmatrix} (11) & (21) & (n1) \\ (12) & (22) & (n2) \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ (1n) & (2n) & (nn) \end{vmatrix} \quad .. \quad (3.1)$$

An element (xx) is given by

$$(xx) = \alpha_x g_x + \sum_y Y_{xy} \quad .. \quad .. \quad (3.2)$$

An element (xy) by

$$(xy) = \alpha_y g_x - Y_{xy} \quad .. \quad .. \quad (3.3)$$

Note that

$$(yx) = \alpha_x g_y - Y_{xy} \neq (xy) \quad .. \quad (3.4)$$

Here Y_{xy} is the admittance of the impedor connecting terminals (x, y), and α_x is the proportion of the total current I_0, permanently leaving the cathode, that arrives at the electrode x. If x is the cathode, α_x is to be taken as -1, so that

$$\sum_x \alpha_x = 0 \quad .. \quad .. \quad .. \quad (3.5)$$

The g's are valve conductances defined by*

$$I_0 = g_2 V_{21} + g_3 V_{31} + \ldots + g_n V_{n1} \quad .. \quad .. \quad (3.6)$$

where the cathode is identified with terminal 1, so that V_{x1} is the potential of terminal x relative to the cathode. g_1 is to be taken as $-$ (sum of the remaining g's), so that

$$\sum_x g_x = 0 \quad .. \quad .. \quad .. \quad .. \quad (3.7)$$

The summation in (3.2, 5, 7) is to be taken over all the terminals including earth; but, since $\alpha_0 = g_0 = 0$, the inclusion of earth does not affect (3.5) or (3.7). Y_{0x} does not occur in (3.3, 4), since neither x nor y in (xy) is 0; but Y_{0x} occurs in (3.2) as part of (xx). It is to be observed that

$$\sum_y (xy) = Y_{0x}; \quad \sum_x (xy) = Y_{0y} \quad .. \quad (3.8)$$

These relations are useful in checking the values assigned to the elements.

If there is no separate earth, but one of the electrodes (say m) is the common terminal of input and output circuits, then Δ reduces to the minor of (mm); all elements that include m disappear; admittances Y_{p0} and Y_{pm}, being now in parallel, are merged. Otherwise nothing is changed. It may be observed that this rule can be obtained by putting $Y_{0m} = \infty$ in the full determinant (3.1).

The general formula for the gains is now

$$A_{kl,\,pq} = Y_{pq} \cdot \rho_{kl,\,pq} \quad .. \quad .. \quad (3.9)$$

where $\rho_{kl,\,pq}$ involves Δ and its cofactors, and depends on Y_{pq}, though it does not contain Y_{pq} as a factor. If the mutual inductances between the impedors are 0 (in which case the amplifier will be said to belong to the class X_1),

$$\rho_{kl,\,pq} = \frac{\Delta_{kp} - \Delta_{kq} - \Delta_{lp} + \Delta_{lq}}{\Delta} \quad (3.10)$$

where Δ_{kl} is the cofactor of (kl) in Δ.

(3.10) will sometimes be written for brevity

$$\rho_{kl,\,pq} = \delta_{kl,\,pq}/\Delta \quad .. \quad .. \quad (3.11)$$

If the mutual inductances are not zero, and the amplifier belongs to class X but not to class X_1, the formula for $\rho_{kl,\,pq}$ is more complicated; the formula when only one pair of impedors have mutual inductances is given in B. But it is to be observed that $\rho_{kl,\,pq}$ has always the dimensions of an impedance. If one of k, l, p, q is 0, two of the Δ's have a suffix 0; since 0 does not appear in any element of Δ, such Δ_{0k} are to be put equal to zero. If one of (k, l) and one of (p, q) is zero (which means that the input and output terminals for this gain have both an earthed terminal), $\delta_{kl,\,pq}$ reduces to one term. If there is no separate earth, but the electrode m is common to input and output, then any Δ which has m as a suffix is to be put equal to zero.

It may be well to insist here that, when a common terminal of input and output is mentioned, *signal* input is always meant. The exceptional position accorded to such a common terminal in formulae which apply to the gain between *any* one pair of terminals and *any* other may appear anomalous. But it is due to a convention adopted in B which is convenient, but not in any way necessary; the formulae could be given in a form that gives no exceptional position to any terminal, but they would be less useful in the problem we are about to discuss and, indeed, in any problem concerning valve

* (3.6) is in effect a definition of what is meant by a linear " single-stage " amplifier.

amplifiers where signal input and output circuits have actually a common terminal.

It has been assumed hitherto that the E.M.F. E_{pq} (which is impedanceless), involved in the definition of the gain in Section 2, is in series with the impedor whose admittance is Y_{pq}, and that there is no other admittance between the terminals (p, q). But in the signal input circuit, this assumption is not usually true; the signal generator is in series with its internal impedance, but the interelectrode capacitance C_{pq} is across the combination. However this

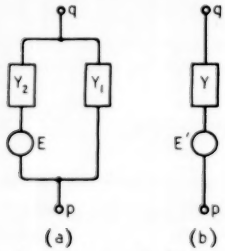

Fig. 1. *By the application of Thévenin's theorem, two admittances (a) can be combined as in (b) if the generator E.M.F. is altered to E'.*

arrangement is equivalent to one of which the assumption is true. For, according to Thévenin's theorem, the arrangement of Fig. 1(*a*) is equivalent to that of Fig. 1(*b*) in respect of any circuit connected to the terminals (p, q) if $Y = Y_1 + Y_2$, and if $E' = EY_2/Y$. Accordingly in such an input circuit, we may replace the parallel impedors by a single impedor whose admittance is the sum of their admittances, if at the same time we replace the E.M.F. E in series with one only of the parallel impedors by an E.M.F. E' in series with the single impedor, where

$$E'Y = EY_2 \quad .. \quad .. \quad .. \quad (3.12)$$

We have then

$$\left.\begin{aligned} V_{kl} &= E'_{pq}\, A_{kl,\,pq} \\ &= E' \,.\, Y \,.\, \rho_{kl,\,pq} \\ &= E \,.\, Y_2 \,.\, \rho_{kl,\,pq} \end{aligned}\right\} \quad .. \quad (3.13)$$

Now $A_{kl,\,pq}$ is used only to calculate V_{kl}; accordingly (3.13) means that, in calculating $A_{kl,\,pq}$, we must use for Y_{pq} the sum of the admittances of all parallel branches of the impedor when we are calculating $\rho_{kl,\,pq}$, but for the factor multiplying $\rho_{kl,\,pq}$ we may use the admittance of that branch which contains the E.M.F., ignoring the other branches, and for E_{pq} the actual E.M.F. in that branch. This is an extremely im-important proposition which will often be used in what follows, sometimes without explicit mention of (3.12).

On the other hand, if, as in Fig. 2, there is an impedor Y_3 in series with both Y_1 and Y_2, in the equivalent Fig. 1(*b*).

$$Y = \frac{Y_3(Y_1 + Y_2)}{Y_1 + Y_2 + Y_3} \quad .. \quad .. \quad (3.14)$$

$$E' = \frac{Y_2}{Y_1 + Y_2} \,.\, E \quad .. \quad .. \quad (3.15)$$

Accordingly in this case, we may *not* substitute EY_2 for $E'Y$ in the factor of $\rho_{kl,\,pq}$, but may substitute $EY_2 \cdot \dfrac{Y_3}{Y_1 + Y_2 + Y_3}$

Another useful formula is that for Y_{kli}, which is the admittance looking into the terminals (k, l). We have, according to B,

$$Y_{kli} = \frac{1}{\rho_{kl,\,kl}} \quad .. \quad .. \quad .. \quad (3.16)$$

If the amplifier is of class X_1, this becomes

$$Y_{kli} = \frac{\Delta}{\delta_{kl,\,kl}} \quad .. \quad .. \quad .. \quad (3.17)$$

4. Formulae for the Common-Grid Triode

We shall now apply the results of Section 3 to the common-grid triode, taking into account the possible presence of unavoidable inductance L in the leads connecting the electrodes to earth. If we identify electrodes

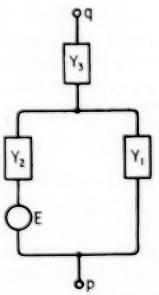

Fig. 2. *Some modification of the formulae is necessary when the admittance Y_3 is included as explained in the text.*

1, 2, 3 respectively with cathode, control grid and anode, the circuit is represented by Fig. 3. The inductance in the grid lead is shown explicitly; but, in order to avoid complications unnecessary at this stage, the inductances in the cathode and anode leads are merged in Y_u, Y_v; they must be taken into account in assigning values to these quantities. E_u is the signal generator (or its

Thévenin equivalent), the admittance of its internal impedance being Y_u. Y_v is the admittance of the load across which the output is taken off. Then we have

$$\left.\begin{aligned} Y_{01} &= Y_u + j\omega C_{01}\,; & Y_{12} &= j\omega C_{12} \\ Y_{02} &= \frac{1}{j\omega L} + j\omega C_{02}\,; & Y_{13} &= j\omega C_{13} \\ Y_{03} &= Y_v + j\omega C_{03}\,; & Y_{23} &= j\omega C_{23} \end{aligned}\right\} \quad (4.1)$$

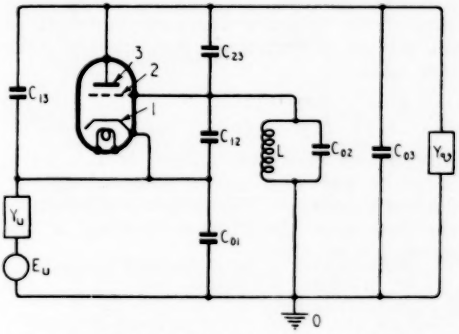

Fig. 3. *The diagram of a common-grid triode amplifier is shown here.*

If the grid is negative and receives no current,

$$\alpha_2 = 0\,; \quad \alpha_3 = 1\,; \quad \alpha_1 = -1 \quad .. \quad (4.2)$$

g_2 is the mutual conductance, g_3 the anode conductance, $g_1 = -(g_2 + g_3)$.

Hence

$$\Delta = \begin{vmatrix} (11) & (21) & (31) \\ (12) & (22) & (32) \\ (13) & (23) & (33) \end{vmatrix}$$

where

$$\left.\begin{aligned} (11) &= g_2 + g_3 + Y_u + j\omega(C_{01} + C_{12} + C_{13}) \\ (21) &= -g_2 - j\omega C_{12} \\ (31) &= -g_3 - j\omega C_{13} \\ (12) &= -j\omega C_{12} \\ (22) &= 1/j\omega L + j\omega(C_{02} + C_{12} + C_{23}) \\ (32) &= -j\omega C_{23} \\ (13) &= -(g_2 + g_3) - j\omega C_{13} \\ (23) &= g_2 - j\omega C_{23} \\ (33) &= g_3 + Y_v + j\omega(C_{03} + C_{13} + C_{23}) \end{aligned}\right\} \quad .. \quad (4.3)$$

The output terminals are (0, 3). If we are concerned only with gains to these terminals, $A_{kl,pq}$ is $A_{03,pq}$, and $\delta_{kl,pq}$ involves only the cofactors Δ_{31}, Δ_{32}, Δ_{33}, which therefore alone need be calculated. Now

$$\left.\begin{aligned} \Delta_{31} &= (12)(23) - (13)(22) \\ \Delta_{32} &= (21)(13) - (23)(11) \\ \Delta_{33} &= (11)(22) - (12)(21) \\ \Delta &= (31)\Delta_{31} + (32)\Delta_{32} + (33)\Delta_{33} \end{aligned}\right\} \quad (4.4)$$

It is not worth while at the present stage to insert values of the elements from (4.4) and multiply out ; for, when numerical values are inserted, many of the numerous terms prove to be negligible.

If $L = 0$ and the grid is really earthed, Δ reduces to the minor of (22) ; *i.e.*

$$\Delta = \begin{vmatrix} (11) & (31) \\ (13) & (33) \end{vmatrix} \quad .. \quad .. \quad (4.5)$$

(2, 3) are the output terminals, and $\delta_{kl,pq}$ involves only

$$\left.\begin{aligned} \Delta_{31} &= -(13) \\ \Delta_{33} &= (11) \end{aligned}\right\} \quad .. \quad .. \quad .. \quad (4.6)$$

where the elements are still given by (4.3).

5. Formulae for the Common-Cathode Pentode

The common-cathode pentode circuit with unavoidable inductance in the earth leads is represented in Fig. 4, the cathode, control grid and anode being again denoted by 1, 2, 3, the screen by 4, and the suppressor by 5. In the same way and for the same reason as before, the inductances in the grid and anode leads are merged in Y_u, Y_v.

Then we have

$$\left.\begin{aligned} Y_{01} &= \frac{1}{j\omega L_1} + j\omega C_{01} \\ Y_{02} &= Y_u + j\omega C_{02} \\ Y_{03} &= Y_v + j\omega C_{03} \\ Y_{04} &= \frac{1}{j\omega L_4} + j\omega C_{04} \\ Y_{05} &= \frac{1}{j\omega L_5} + j\omega C_{05} \end{aligned}\right\} \quad .. \quad (5.1)$$

otherwise

$$Y_{pq} = j\omega C_{pq}$$

α_3, α_4 are the proportions of the current received by anode and screen

$$\alpha_3 + \alpha_4 = 1\,; \quad \alpha_2 = \alpha_5 = 0\,; \quad \alpha_1 = -1$$
$$g_1 = -g_2 - g_3 - g_4 - g_5$$
$$.. \quad .. \quad (5.2)$$

Then

$$\Delta = \begin{vmatrix} (11) & (21) & (31) & (41) & (51) \\ (12) & (22) & (32) & (42) & (52) \\ (13) & (23) & (33) & (43) & (53) \\ (14) & (24) & (34) & (44) & (54) \\ (15) & (25) & (35) & (45) & (55) \end{vmatrix} \quad .. \quad .. \quad (5.3)$$

$$(11) = -g_1 + \frac{1}{j\omega L_1} + j\omega(C_{01} + C_{12} + C_{13} + C_{14} + C_{15})$$
$$(21) = -g_2 - j\omega C_{12}$$
$$(31) = -g_3 - j\omega C_{13}$$
$$(41) = -g_4 - j\omega C_{14}$$
$$(51) = -g_5 - j\omega C_{15}$$
$$(12) = -j\omega C_{12}$$
$$(22) = Y_u + j\omega(C_{02} + C_{12} + C_{23} + C_{24} + C_{25})$$
$$(32) = -j\omega C_{23}$$
$$(42) = -j\omega C_{24}$$
$$(52) = -j\omega C_{25}$$
$$(13) = \alpha_3 g_1 - j\omega C_{13}$$
$$(23) = \alpha_3 g_2 - j\omega C_{23}$$
$$(33) = \alpha_3 g_3 + Y_v + j\omega(C_{03} + C_{13} + C_{23} + C_{34} + C_{35})$$
$$(43) = \alpha_3 g_4 - j\omega C_{34}$$
$$(53) = \alpha_3 g_5 - j\omega C_{35}$$
$$(14) = \alpha_4 g_1 - j\omega C_{14}$$
$$(24) = \alpha_4 g_2 - j\omega C_{24}$$
$$(34) = \alpha_4 g_3 - j\omega C_{34}$$
$$(44) = \alpha_4 g_4 + \frac{1}{j\omega L_4} + j\omega(C_{04} + C_{14} + C_{24} + C_{34} + C_{45})$$
$$(54) = \alpha_4 g_5 - j\omega C_{45}$$
$$(15) = -j\omega C_{15}$$
$$(25) = -j\omega C_{25}$$
$$(35) = -j\omega C_{35}$$
$$(45) = -j\omega C_{45}$$
$$(55) = \frac{1}{j\omega L_5} + j\omega(C_{05} + C_{15} + C_{25} + C_{35} + C_{45})$$
$$.. \quad .. \quad (5.4)$$

If the inductances L_1, L_4, L_5 are all zero, only those elements that are common to the minors of (11), (44), (55) survive. We then have

$$\Delta = \begin{vmatrix} (22) & (32) \\ (23) & (33) \end{vmatrix} \quad .. \quad .. \quad (5.5)$$

Any Δ, one of whose suffixes is 1, 4 or 5, is to be put equal to 0. Accordingly α_4 does not appear. This is the expression of the fact that the current flowing to the screen has no effect.

The output terminals are (1, 3) and $\delta_{kl,pq}$ involves only

$$\left.\begin{aligned} \Delta_{32} &= -(23) \\ \Delta_{33} &= \quad (22) \end{aligned}\right\} \quad .. \quad .. \quad (5.6)$$

These results are used in section 12.

6. Important Relations for an Impedor

We also need some subsidiary propositions. One set concerns well-known properties of impedors. Let Z, Y, R, X, G, B be respectively the impedance, admittance, resistance, reactance, conductance and susceptance of an impedor; this notation will

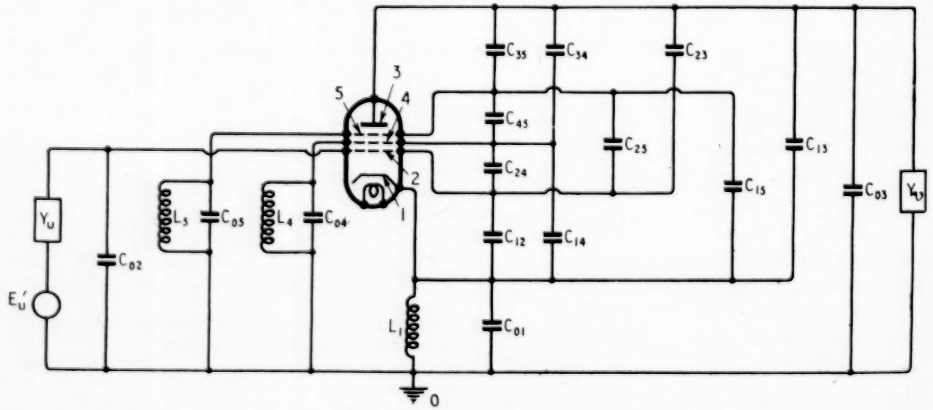

Fig. 4. *The common-cathode pentode circuit, when account is taken of the lead inductances and inter-electrode capacitances.*

be employed hereafter without notice, e.g. R_x will always mean the real part of Z_x.

Then

$$\left.\begin{aligned} Y = G + jB = \frac{1}{Z} = \frac{1}{R + jX} \quad \text{(a)} \\ R|Y|^2 = G \text{ (b)}; \quad X|Y|^2 = -B \text{ (d)} \\ G|Z|^2 = R \text{ (c)}; \quad B|Z|^2 = -X \text{ (e)} \end{aligned}\right\} \quad (6.1)$$

Frequent use will be made of 6.1(b); the reader should bear it in mind.

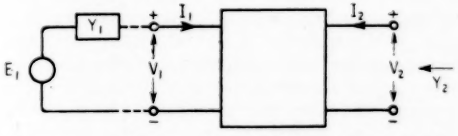

Fig. 5. *Defining voltages and currents used in equations for perfect transformer in equation 7.1 of text.*

7. General Properties of a Transformer

The other set concerns the properties of 4-terminal passive networks which we shall term transformers. When the elements of the network are all pure reactors and there is no dissipation, the network will be called a *perfect* transformer (*PT*).

If I_1, I_2, V_1, V_2 are the currents into and voltages across the terminals of a *PT* (see Fig. 5),

$$\left.\begin{aligned} I_1 = jy_{11}V_1 + jy_{12}V_2 \\ I_2 = jy_{12}V_1 + jy_{22}V_2 \end{aligned}\right\} \quad .. \quad .. \quad (7.1)$$

where the y's are all real. If an E.M.F. E_1, in series with an impedor of admittance Y_1, is connected across the input terminals, and if the output terminals are open, a voltage E_2 will appear across the output terminals and the admittance looking into them will have some value Y_2; by Thévenin's theorem that arrangement is then equivalent, in respect of anything connected to the terminals 2, to an E.M.F. E_2 in series with an impedor of admittance Y_2. We require a relation between E_1, E_2, Y_1, Y_2. (It should be noted that, though Y_2 is the admittance looking into terminals 2, Y_1 is *not* the admittance looking into terminals 1; for the admittance of the network is in parallel with Y_1. This asymmetry is natural to the particular problem that concerns us.)

E_2 can be found by combining (7.1) with

$$V_1 = E_1 - I_1/Y_1; \quad I_2 = 0; \quad V_2 = E_2 \quad .. \quad .. \quad (7.2)$$

This leads to

$$\frac{E_2}{E_1} = \frac{-y_{12}Y_1}{y_{22}Y_1 - j(y_{12}^2 - y_{11}y_{22})} \quad .. \quad (7.3)$$

The admittance Y_2 can be obtained by making $E_1 = 0$ and introducing an E.M.F. between the terminals 2. The equations to be combined with (7.1) are then

$$I_1 = -V_1Y_1; \quad I_2 = V_2Y_2 \quad .. \quad (7.4)$$

This leads to

$$Y_2 = \frac{y_{12}^2 - y_{11}y_{22} + jy_{22}Y_1}{Y_1 + jy_{11}} \quad .. \quad (7.5)$$

Writing $Y_1 = G_1 + jB_1$, $Y_2 = G_2 + jB_2$, we find

$$G_2 = G_1\frac{y_{12}^2}{G_1^2 + (B_1 + y_{11})^2} \quad .. \quad (7.6)$$

and

$$\left|\frac{E_2}{E_1}\right|^2 = \frac{y_{12}^2|Y_1|^2}{|Y_2|^2[G_1^2 + (B_1 + y_{11})^2]} = \left|\frac{Y_1}{Y_2}\right|^2 \cdot \frac{G_2}{G_1} \quad .. \quad .. \quad (7.7)$$

$$[\text{by 6.1 (b)}] = \frac{R_2}{R_1} \quad .. \quad .. \quad (7.8)$$

If the transformer always operates at the same frequency, the three y's in (7.1) are in principle completely at our disposal. If, given Y_1, G_2 and B_2 are to have assigned values, (7.5) gives two equations between them; the third is still free. Accordingly in principle we can give Y_2 any assigned value (subject to the limitation that, if G_1 is positive, so must be G_2), and still have a characteristic of the *PT* at our disposal. In practice our freedom of choice is not so great. If the transformer is a uniform line closed at one end with tappings for the input and output, two of the characteristics are the distances of these tappings from the closed end, which can easily be made adjustable. But the third is then the characteristic impedance of the line, which is not so easily adjustable.

No actual transformer is perfect; it always has some dissipation. The y's in (7.1) are then complex and, since the network is passive, the real part of each jy will be positive. Then, even if the three y's are completely at our disposal, subject to this limitation, it is not possible to give Y_2 any desired value; in particular G_2, the real part of Y_2, will be non-zero and positive. We shall assume that, in considering the adjustment of an imperfect transformer at an

assigned frequency, we may regard it as the equivalent of a PT with a conductor of conductance γ across its output terminals. We must not assume that γ is independent of the adjustment of the PT or that, when the PT has been adjusted to fulfil some condition, γ is independent of frequency, but only that, when the PT has been adjusted and operates at an assigned frequency, γ has some ascertainable value. We do not pretend that this assumption can be proved rigidly to be true generally; but it appears to be highly plausible in the limited conditions in which we shall actually use it.

8. Further Development of the Noise Formulae

We now start to combine the very general propositions that have been enunciated to give formulae applicable to our problem. v^2 with suffixes will always denote a component (or possibly the whole) of the noise $\overline{(v-\bar{v})^2}$; a suffix θ will denote that the component is thermal noise, a suffix σ that it is shot noise.

Combining (2.3) and (3.10), we have for the component of the noise at the output terminals (k, l) of a stage, arising from the impedor between the terminals (p, q),

$$v^2_{\theta pq} = \frac{2k'T}{\pi}\int_0^\infty R_{pq}|Y_{pq}|^2|\rho_{kl,pq}|^2\,d\omega = 4k'T\int_0^\infty G_{pq}|\rho_{kl,pq}|^2\,df \quad .. \quad (8.1)$$

If an impedor contains several resistors, it is often important, *e.g.* when one of the resistors is the signal generator, to distinguish the contributions of the different resistors. If the resistors lie in different independent parallel branches, and G_{pq1}, G_{pq2}, . . . are the conductances of these branches, then the appropriate formula is

$$v^2_{\theta pq} = 4k'T\int_0^\infty (G_{pq1}+G_{pq2}+\ldots)|\rho_{kl,pq}|^2 df \quad .. \quad .. \quad (8.2)$$

Of course, in calculating $\rho_{kl,pq}$ all the branches are taken into account.

(8.2) is a consequence of the general proposition that a conductor consisting of several branches is *in all respects* equivalent to a single conductor whose conductance is the sum of those of its branches. In order to deduce it more particularly, we have to go behind (2.3) and consider the E.M.F.'s that simulate the noise events. Then (8.2) appears as a consequence of (3.13).

The capacitor, across which electrons pass through the vacuum between a pair of terminals, cannot have anything in series with it, though the impedor connecting those terminals may include elements in parallel with the capacitor. Accordingly the admittance of that branch of the impedor which is in series with the E.M.F. simulating the shot noise is simply that of the capacitor, *i.e.* $j\omega C_{pq}$. Combining (2.4) and (3.10), we have therefore for the component of the noise at the output terminals (k, l) due to the space current between the terminals (p, q), when there is no partition and all the electrons leaving p arrive at q,

$$v^2_{\sigma pq} = \frac{I_0\epsilon\Gamma^2}{\pi}\int_0^\infty \frac{1}{|Y_{pq}|^2}\cdot|Y_{pq}|^2|\rho_{kl,pq}|^2\,d\omega = 2I_0\epsilon\Gamma^2\int_0^\infty|\rho_{kl,pq}|^2\,df \quad .. \quad .. \quad (8.3)$$

If we write

$$2\epsilon/4k'T = a \quad .. \quad .. \quad .. \quad (8.4)$$

this becomes

$$v^2_{\sigma pq} = 4k'T\int_0^\infty aI_0\Gamma^2|\rho_{kl,pq}|^2\,df \quad .. \quad .. \quad (8.5)$$

a has the dimensions of (voltage)$^{-1}$, and at room temperature (292 deg. K) is 20 volt^{-1}. $I_0\Gamma^2 a$ has the dimensions of a conductance. If we denote it by $G_{pq\sigma}$, the combined thermal and shot noise arising between the terminals (p, q) takes the symmetrical form

$$v^2_{pq} = 4k'T\int_0^\infty (G_{pq}+G_{pq\sigma})|\rho_{kl,pq}|^2\,df \quad .. \quad .. \quad (8.6)$$

The whole noise at the output terminals is then

$$v^2 = 4k'T\int_0^\infty \sum_{pq}(G_{pq}+G_{pq\sigma})\,|\rho_{kl,pq}|^2\,df \quad .. \quad .. \quad (8.7)$$

the summation being taken over all pairs of terminals (p, q).

If the electron stream leaving terminal p is divided between terminals q, r, we have from (2.5) and (3.10), that the noise output, which we denote by $v^2_{\sigma pqr}$ is given by

$$v^2_{\sigma pqr} = 2I_0\epsilon \int_0^\infty [\alpha_q(\alpha_q\Gamma^2 + \alpha_r)\,|\rho_{kl,pq}|^2 + \alpha_r(\alpha_r\Gamma^2 + \alpha_q)|\rho_{kl,pr}|^2 - 2\alpha_q\alpha_r(1-\Gamma^2)\{\text{real part of}\,\rho_{kl,pq}|\rho^*_{kl,pr}\}]\,df \quad .. \quad .. \quad (8.8)$$

If we write

$$\rho_{kl,pq} = \eta_1 + j\xi_1\,; \quad \rho_{kl,pr} = \eta_2 + j\xi_2 \quad .. \quad .. \quad (8.9)$$

this becomes

$$v^2_{\sigma p,qr} = 4k'T\int_0^\infty aI_0[\Gamma^2\{(\alpha_q\eta_1 + \alpha_r\eta_2)^2 + (\alpha_q\xi_1 + \alpha_r\xi_2)^2\} + \alpha_q\alpha_r\{(\eta_1 - \eta_2)^2 + (\xi_1 - \xi_2)^2\}]\,df \quad .. \quad .. \quad (8.10)$$

$$= 4k'T\int_0^\infty aI_0[\Gamma^2|\alpha_q\rho_{kl,pq} + \alpha_r\rho_{kl,pr}|^2 + \alpha_q\alpha_r|\rho_{kl,pq} - \rho_{kl,pr}|^2]\,df \quad .. \quad .. \quad (8.11)$$

If $\rho_{kl,pq} = \rho_{kl,pr}$, the second term of the integrand vanishes and, since $\alpha_q + \alpha_r = 1$, (8.11) reduces to (8.5), as it should.

9. Stages in Cascade

Equations (8.2, 3, 4, 5, 8, 11) are true of all class X amplifiers, whether they are or are not of class X_1. In each of them the integrand has the dimensions of an impedance. Accordingly in class X amplifiers any component of the noise at the output terminals, or the whole noise, which is the sum of these components, can be written in the familiar form,

$$v^2 = 4k'T\int_0^\infty Z\,df \quad .. \quad .. \quad (9.1)$$

We now write the noise due to the rth stage at the output terminals of the rth stage as

$$v_r^2 = 4k'T\int_0^\infty Z_r\,df \quad .. \quad .. \quad (9.2)$$

where in general Z_r is equal to the integrand of (8.11) summed over all pairs of terminals p_q plus the sum of the thermal noise terms corresponding to the $\sum_{pq} G_{pq}|\rho_{kl,pq}|^2$ of (8.7).

If there is no partition noise Z_r reduces to the integrand of (8.7).

The noise at the output terminals of the amplifier due to that arising in the rth stage is

$$4k'T\int_0^\infty Z_r|A_{r+1}|^2|A_{r+2}|^2 \ldots df \quad .. \quad .. \quad .. \quad .. \quad (9.3)$$

In (9.3) A_{r+1} is the gain of the $(r+1)$th stage defined as follows (see Fig. 6): so

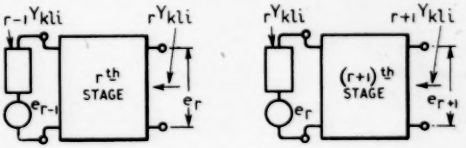

Fig. 6. *This diagram defines the parameters of the rth and (r + 1)th stages.*

far as the effect of the rth stage on the $(r+1)$th and succeeding stages is concerned the rth stage can be represented by an impedanceless voltage generator of frequency f, in series with an admittance equal to the output admittance of the rth stage which we shall denote by ${}_rY_{kli}$. (This notation is general. When a suffix r denoting a stage is attached to a symbol with other suffixes, r will be placed before the symbol.) Then if the generator of voltage $e_r(f)$ produces a voltage at the open circuit output terminals of $e_{r-1}(f)$ we define A_{r+1} as

$$A_{r+1} = \frac{e_{r+1}}{e_r} \quad .. \quad .. \quad .. \quad (9.4)$$

In the case of the first stage the gain A_1 will be the open circuit voltage of the first stage divided by the E.M.F. of the signal generator.

It should be noted here that the network connecting two stages can be divided into two parts at any point; one part of the network is regarded as part of the first stage, the output terminals of which then become those at which the network is cut. The other part is considered as part of the succeeding stage. This is considered in more detail in Section 11.

(To be concluded.)

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TRANSIENT RESPONSE OF FILTERS

By D. G. Tucker, Ph.D., A.M.I.E.E.

(Post Office Research Station)

(Concluded from page 42 of the February issue)

3. Single-Section Filters with Resistance-Terminations. Analysis Based on the Actual Circuit

FOR some purposes it is necessary to assess the performance of single-section filters, and for such cases the two methods described above are not very suitable. The ideal filter method gives considerable errors, whilst the Carson and Zobel method is too complicated for ordinary use, as well as being based essentially on image terminations. A single-section filter is normally used between resistance terminations equal to its design resistance R. The following work shows some of the properties of such circuits, worked out where necessary by the use of the Heaviside Expansion Theorem.

The only type of section considered is the series-derived T arrangement, of "symmetrical" 4- or 6-element band-pass type. As far as transient response is concerned, this is probably sufficiently representative of all symmetrical types. The steady-state responses, and particularly the effects of dissipation, are considerably different in the various cases, however; this is discussed fairly fully in a previous paper[11].

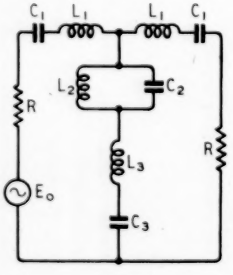


Fig. 3. *6-element symmetrical filter.*

3.1 *Analysis of "6-element Symmetrical" Filter Section*

The circuit diagram of this filter is shown in Fig. 3. The arrangement is the series-derived T-section.

If m is the "derivation" parameter,

n = fractional bandwith

$$= \frac{\text{actual bandwidth}}{\text{mid-band frequency}}$$

R = design impedance

Then

$$L_1 = \frac{m}{n} \cdot \frac{R}{\omega_0} \qquad C_1 = \frac{n}{m} \cdot \frac{1}{R\omega_0}$$

$$L_2 = \frac{n}{2m} \cdot \frac{R}{\omega_0} \qquad C_2 = \frac{2m}{n} \cdot \frac{1}{R\omega_0}$$

$$L_3 = \frac{1-m^2}{2mn} \cdot \frac{R}{\omega_0} \qquad C_3 = \frac{2mn}{1-m^2} \cdot \frac{1}{R\omega_0}$$

Now if the filter is replaced by the simple network of Fig. 4, we have

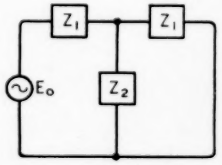


Fig. 4. *Simplified network representing Fig. 3.*

$$\frac{\text{Output current}}{E_0} = \frac{Z_2}{Z_1(Z_1 + 2Z_2)} \quad \ldots \quad (14)$$

In the actual circuit,

$$Z_1 = R\left(1 + \frac{mp}{n\omega_0} + \frac{m\omega_0}{np}\right),$$

and

$$Z_2 = R\left[\frac{\frac{pn}{2m\omega_0}}{1 + p^2/\omega_0^2} + \frac{p(1-m^2)}{2mn\omega_0} + \frac{(1-m^2)\omega_0}{2mnp}\right]$$

where $p = \frac{d}{dt}$ = differential operator;

$$\frac{1}{p} = \int_0^t \ldots dt$$

Putting $\frac{m}{n}\left(\frac{p}{\omega_0} + \frac{\omega_0}{p}\right) = M$

$$Z_1 = R(1 + M)$$

and $$Z_2 = \frac{R}{2}\left(\frac{1}{M} + \frac{1-m^2}{m^2} M\right)$$

$\therefore$ from (14), $\frac{\text{Output volts}}{\frac{1}{2}E_0}$

$$= \frac{1 + \frac{1-m^2}{m^2} M^2}{1 + 2M + \frac{1+m^2}{m^2} M^2 + \frac{M^3}{m^2}} \quad \ldots \quad \ldots \quad (15)$$

3.11 *Build-up and Decay Curves for Applied Frequency = Mid-band Frequency*

The correct way to determine these would be by the use of the Heaviside Expansion Theorem [12, 13]. Equation (15) would be written

$$\frac{\text{Output volts}}{\frac{1}{2}E_0} = \frac{Y_{(p)}}{Z_{(p)}}\mathbf{1} \quad .. \quad .. \quad (16)$$

where $Y_{(p)}$ and $Z_{(p)}$ are expressed as functions of p and $\mathbf{1}$ is the Heaviside Unit Function.

Then the roots would be found of

$$Z_{(p)} = 0 \quad .. \quad .. \quad .. \quad .. \quad (17)$$

and if these are p_1, p_2, p_3 etc., then the solution of (16) is

$$\frac{Y_{(0)}}{Z_{(0)}} + \sum_{p=p_1}^{p=p_n} \frac{Y_{(p)}\epsilon^{pt}}{p\dfrac{dZ_{(p)}}{dp}} \quad .. \quad .. \quad (18)$$

Since the applied E.M.F. is alternating (i.e. $E_0 \sin \omega t$),

$$\frac{Y_{(p)}}{Z_{(p)}} = \frac{p\omega}{p^2+\omega^2} \times \text{the original right-hand side of equation (15)}$$

Since $p^2 + \omega^2$ is now a factor of $Z_{(p)}$, one solution of $Z_{(p)} = 0$ is $p = \pm j\omega$. This gives the steady-state term in (18), for the exponentials concerned are

$$\epsilon^{+j\omega t} + \epsilon^{-j\omega t} \text{ (with the same modulus)}$$

which combine to give $2 \cos \omega t$.

The other solutions of $Z_{(p)} = 0$ give the transient terms, and if these are oscillatory, the roots of $Z_{(p)}$ occur in conjugate pairs; the real part gives the decay exponential index and the imaginary part gives the frequency.

This working may only be applied to the amplitude terms if no roots of $Z_{(p)}$ are equal or zero.

Now this method does not work out satisfactorily for complicated circuits such as filters, for in such cases $Z_{(p)}$ is a polynomial of high order, and it is practically impossible to find its roots. In some cases, $Z_{(p)}$ can be expressed in some simple functional form, and the solutions for such cases have been worked out and tabulated by Campbell and Foster[14]. But in the present instance the denominator of (15) is a sixth-power polynomial in p, and no such solution is available.

A method of dealing with these complicated circuits has been described by Laurent[15, 16], under the general title of "Frequency Transformations." The process is to replace the complex network by a simple one which is identical when multiplied throughout by a frequency transformation function. The equation $Z_{(p)} = 0$ then refers only to the simple network, and the final transient expression has additional amplitude and exponential factors. It will generally be fairly straightforward to solve $Z_{(p)} = 0$. The method has been tried for the 6-element filter, but the result obtained by the author for the amplitude terms is not correct.

In view of the difficulties described above, no analysis of the transient amplitude response is given here. But the build-up characteristics can be dealt with on a largely analytical basis.

The *natural frequencies of the system* must first be found. These can be determined fairly closely by putting the terminating resistances equal to zero; the equations are then sufficiently simplified to enable a solution to be obtained. So we have

$$Z_1 = RM$$

$$Z_2 = \frac{R}{2}\left(\frac{1}{M} + \frac{1-m^2}{m^2}M\right)$$

And from (14)

$$\frac{\text{Output volts}}{\frac{1}{2}E_0} = \frac{\dfrac{1}{M} + \dfrac{1-m^2}{m^2}M}{M\left(\dfrac{1}{M} + \dfrac{M}{m^2}\right)}$$

For the transient terms we must find the roots of

$$M\left(\frac{1}{M} + \frac{M}{m^2}\right) = 0$$

i.e. $$M^2 + m^2 = 0$$

i.e. $$\left(\frac{p}{\omega_0} + \frac{\omega_0}{p}\right)^2 = -n^2$$

$$\therefore \quad p = \pm j\omega_0\sqrt{1 + \frac{n^2}{2} \mp n\sqrt{1+\frac{n^2}{4}}}$$

Also $M = 0$

$$\therefore \quad p = \pm j\omega_0$$

Thus, if the bandwidth, n, is small, we may say

$$\text{Transient frequencies} = \omega_0,\ \omega_0\left(1 + \frac{n}{2}\right) \text{ and } \omega_0\left(1 - \frac{n}{2}\right)$$

There are, of course, no decay (i.e. real)

indices, as the terminating resistances are zero. We can assume that the transient frequencies are unaffected by the terminating resistances if the bandwidth is small, for then $R \ll \omega_0 L$, etc.

To obtain the decay index we must solve a more complicated equation, i.e. the denominator of (15),

$$\text{i.e.} \quad 1 + 2M + \frac{1+m^2}{m^2} M^2 + \frac{M^3}{m^2} = 0$$

$$\text{i.e.} \quad y^6 + \frac{n}{m}(1+m^2)y^5 + (2n^2+3)y^4 + \left[\frac{n^3}{m} + \frac{2n}{m}(1+m^2)\right] y^3 + (2n^2+3)y^2 + \frac{n}{m}(1+m^2)y + 1 = 0$$

where $y = p/\omega_0$.

If we neglect terms in n^2 and n^3 in comparison with unity

$$y^6 + \frac{n}{m}(1+m^2)y^5 + 3y^4 + \frac{2n}{m}(1+m^2)y^3 + 3y^2 + \frac{n}{m}(1+m^2)y + 1 = 0 \quad (19)$$

This is very nearly

$$\left[y^2 + \frac{n}{3m}(1+m^2)y + 1\right]^3 = 0$$

which gives, very nearly (neglecting n^2 terms)

$$y = -\frac{n(1+m^2)}{6m} \pm j$$

$$\text{i.e.} \quad \text{Decay index} = \frac{-n(1+m^2)\omega_0}{6m} t$$

(with a frequency $\doteqdot \omega_0$—but this is too inaccurate, hence the preceding work with zero terminating resistances).

Now we have a transient consisting of three frequencies, ω_0 and $\omega_0(1 \pm \frac{n}{2})$. If the phases are suitable, these represent an amplitude modulated wave, which can be expressed as

$$\sin \omega_0 t \left(1 + G \sin \frac{n\omega_0}{2} t\right)$$

where G is a constant. Moreover, for an applied frequency equal to the mid-band frequency, the transient and steady-state components must be equal and opposite at $t = 0$, and we may therefore not unreasonably express the build-up envelope characteristic as

$$1 - \epsilon^{\frac{-n(1+m^2)\omega_0 t}{6m}} \cdot \left(1 + G \sin \frac{n\omega_0}{2} t\right) \quad (20)$$

and the decay as the same function without the $(1-)$ term.

If fewer approximations are made, the result may be obtained in a somewhat different form.

Equation (19) may be written

$$\left[y^2 + \frac{n}{m}y + 1\right]\left[y^4 + mny^3 + (2+n^2)y^2 + mny + 1\right] = 0$$

$$\text{i.e.} \quad \text{(a)} \quad y^2 + \frac{n}{m}y + 1 = 0$$

$$\text{i.e.} \quad y = -\frac{n}{2m} \pm j$$

$$\text{(b)} \quad y^4 + mny^3 + (2+n^2)y^2 + mny + 1 = 0$$

An expression, obtained by the use of the Laurent Frequency Transformations, which can be seen to fit very closely to this equation, is

$$y = -\frac{mn}{4} \pm j\left(1 \pm \frac{n}{2}\sqrt{1 - \frac{m^2}{4}}\right)$$

Here, then, we have from (a), the frequency ω_0 with an index $-\frac{n\omega_0 t}{2m}$; and from (b) the frequencies $\omega_0\left(1 \pm \frac{n}{2}\sqrt{1-\frac{m^2}{4}}\right)$ with an index $\frac{-mn\omega_0 t}{4}$*. This leads to a build-up curve thus:—

$$1 - \epsilon^{-\frac{n\omega_0 t}{2m}}\left[1 + G\epsilon^{\left(-\frac{mn}{4} + \frac{n}{2m}\right)\omega_0 t} \cdot \sin \frac{n\omega_0}{2}\sqrt{1-\frac{m^2}{4}} \cdot t\right] \quad (21)$$

The build-up curve of (20) is rather simpler to compute, but is not such a good match to the measured values as (21), as will be seen from the examples which follow. The constant G can theoretically be determined from the Expansion Theorem, but this has not been found feasible by direct processes. The analogous T-section of low-pass filter, with a D.C. pulse applied, can, however, be

* It is worth noting that in the case of the low pass filter analogous to the band-pass being considered, the natural frequencies can readily be shown to be *exactly* 0 and $\omega_c\sqrt{1-\frac{m^2}{4}}$, which correspond correctly to those given in equation (21) according to the rules given in the introduction.

solved fairly readily, and gives values of G as follows :—*

$m = 1.0$, $G = 1.15$
$m = 0.9$, $G = 1.0$
$m = 0.8$, $G = 0.85$

The value of G for $m = 1.0$ appears to apply fairly closely to the band-pass case, but experiments (described below) indicate that, with dissipative narrow-band filters at least, the values of G for $m = 0.9$ and 0.8 should be somewhat lower.

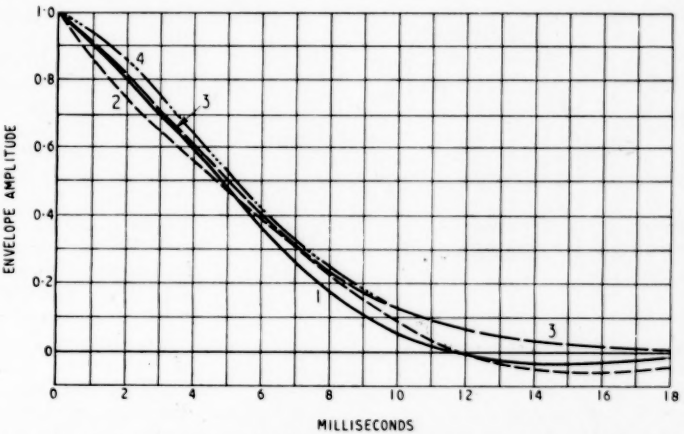


Fig. 5. *Decay curves for* $n = 0.037$, $m = 0.884$, $\omega_0 = 16.65 \times 10^3$. *Curve* 1, *measured; curve* 2, *calculated from* $\epsilon^{-0.349t}(1 + 0.6\ \epsilon^{0.212t} \sin 0.276t)$; *curve* 3, *calculated from* $\epsilon^{0.207t}(1 + 0.4 \sin 0.308t)$; *curve* 4, *calculated from* $\epsilon^{-0.207t}(1 + 0.5 \sin 0.308t)$.

It must be appreciated that equations (20) and (21) are empirical, since it has been assumed that the phases of the components are suitable; *but the ideal filter theory gives an amplitude modulated wave (see equations* 5 *and* 6) *so it is not unreasonable to expect the same in this case.*

An example and a check against measured results will now be discussed. The filter concerned had $m = 0.884$, $n = 0.037$ and $\omega_0 = 16.65 \times 10^3$ and the Q value of the inductors was 100. The unit of time was taken as 1 ms, so that the equations of decay (which is easier to consider than build-up) are

(*a*) corresponding to eqn. (20) :—
$\epsilon^{-0.207t} \cdot (1 + G \sin 0.308t)$

(*b*) corresponding to eqn. (21) :—
$\epsilon^{-0.349t}(1 + G\ \epsilon^{0.212t} \cdot \sin 0.276t)$

Fig. 5 shows, in Curve 1, the characteristic measured on the actual filter by means of a Cambridge Oscillograph. The curve shown is actually taken from the build-up, but the measured decay curves are very nearly the same, of course. It will be seen that the envelope becomes zero at one place, and actually is oscillatory, with high damping. This is generally found on the decay, but the corresponding effect of overshoot is not always observable on the build-up. Curve 3 is the simpler equation (20), with $G = 0.4$, and gives a fair match, but does not show the oscillatory or overshoot effect. The same equation with $G = 0.5$ is shown in Curve 4, so that the effect of a change in G can be observed. Curve 2 shows the more complex equation (21), with $G = 0.6$. This shows the overshoot, and on the whole gives the best match of all. It should be noted, though, that the build-up time measured at the half-amplitude point (according to the definition) is given more accurately by the simpler equation (20); the slope of (21) at half-amplitude is considerably different from the measured value.

It is seen that the G values given earlier are not necessarily the best, and that rather lower values of G are more suitable in practice. This may be the effect of dissipation in the filter, which cannot easily be allowed for.

The above work enables the build-up and decay curves to be calculated with enough accuracy for ordinary purposes. It has the disadvantage that it does not lead to a simple expression of the rate of build-up at half amplitude. But it will be observed that

* For the π-section of low-pass filter with $m = 1$, Kallmann, Spencer and Singer [19] obtain the following build-up expression (see their equation 11) :—

$1 - \epsilon^{-\omega_0 t} - 1.15\ \epsilon^{-\omega_0 t/2} \cdot \sin 0.86\ \omega_c t$

It will be seen that this is identical with the results obtained in the present paper for the T-section, with G equal to 1.15. This confirms an earlier suggestion that the transient responses of the various types of section are more or less the same.

the rate deduced for the idealised filter—see equation (4*b*) applies quite closely in practice to the single-section filter. The filter discussed in the above example had a bandwidth of 100 c/s, and a measured build-up at the half-amplitude rate of about 9.5 ms, as compared with 10 ms deduced by equation (4*b*).

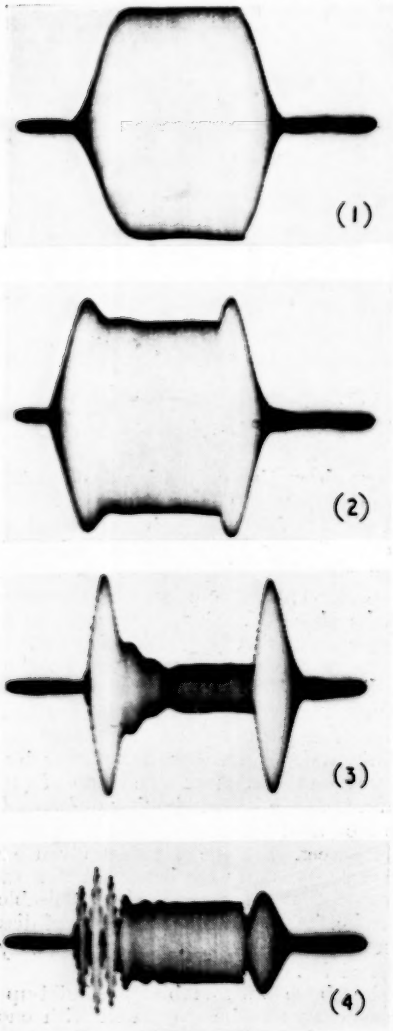


Fig. 6. *Pulse waveforms in 6-element filter section (bandwidth = 100 c/s); applied frequencies,* (1) 2,650 *c/s = mid-band,* (2) 2,600 *c/s,* (3) 2,550 *c/s,* (4) 2,400 *c/s.*

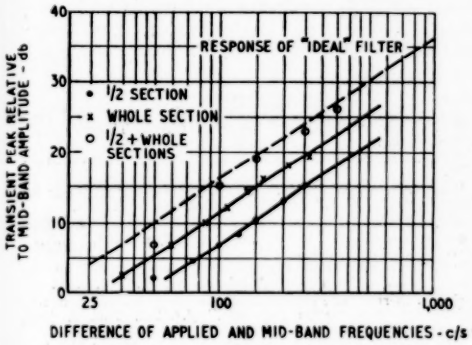
3.12 *Transient Interference Response*

The question of the peak amplitude of the transient signal at the output of the filter, when a pulse of frequency outside the band is applied to the filter input, is a rather difficult one. The ideal filter theory discussed in section 1.1 gives a very simple relationship (equation 6) which is useful in practice for filters of 1½ or 2 sections or more. For the single-section filter, the transient peak amplitude is larger than in the ideal filter, but it has already been explained that the analysis is difficult and is not dealt with here. Consequently only some measured results will be given and discussed.

Fig. 6 shows the waveform of the signals at various frequencies. All were photographed from a Cathode-ray Oscillograph, and the relative amplitudes are chosen only for convenience, having no other significance. The filter concerned was the 6-element whole-section previously discussed. It had a 100 c/s bandwidth with a mid-band frequency of 2,650 c/s. No. 1 shows the mid-band build-up and decay curves, which have been discussed in the previous section. No. 2 shows how the waveform appears when the applied frequency is one of the cut-off frequencies. The transient already exceeds the steady-state component. This is almost always true of filters. Very rarely can the steady-state amplitude at any frequency exceed the transient. No. 3 shows the waveform when the applied frequency is at the frequency of the attenuation (steady-state) peak. Here the steady-state component is almost negligible. No. 4 shows the waveform at a frequency where the steady-state attenuation is near its minimum value, and the transient and steady-state components are nearly equal. Owing to the phasing in and out of the two components during the "make" build-up period, the maximum amplitude of the signal is nearly twice that of the transient component.

As regards the amplitudes of the transient peaks, Fig. 1 shows these compared with those of an ideal filter (calculated from equation 6) and also with those measured on a half-section filter and on the combination of whole plus half-sections. It will be noted that the transient peak amplitudes of a 1½-section filter are near enough to those of the ideal filter, and it has been found that this is generally true for narrow-band filters of more than one section of 6-element or constant-*k* type. The transient peaks of the

single section are, however, 5 db larger than in the ideal filter. For design purposes it should be satisfactory to state that the transient peak amplitudes of a single-section



This diagram is a repetition of Fig. 1, showing the transient response of 6-element filters; bandwidth = 100 c/s, mid-band = 2,650 c/s.

6-element filter are twice those of the ideal filter of the same design bandwidth. In addition, some allowance must be made for the effect of the steady-state component adding to the transient at certain points in the frequency range.

3.2 *Analysis of Constant-k Filter Section*

The constant-k filter is merely a special case of the 6-element filter when $m = 1$. Consequently the equations of the previous section apply, and become as follows :—

Build-up envelope according to the simpler theory (see equation 20)

$$1 - \epsilon^{\frac{-n\omega_0 t}{3}}\left(1 + G \sin \frac{n\omega_0}{2} t\right) \quad .. \quad (22)$$

and according to the more complex theory (see equation 21) by

$$1 - \epsilon^{\frac{-n\omega_0 t}{2}}\left(1 + G\epsilon^{\frac{+n\omega_0 t}{4}} . \sin \frac{n\omega_0}{2}\sqrt{\tfrac{3}{4}} . t\right) \quad .. \quad .. \quad (23)$$

A single-section constant-k filter was made and tested. It had $\omega_0 = 16.03 \times 10^3$ (i.e. 2,550 c/s mid-band), $n = 0.038$ (i.e. 100 c/s bandwidth) and the Q of the inductors was about 100.

The measured build-up and decay curves were found to be slightly different, in that the build-up showed no " overshoot." Consequently both are shown on the diagram, Fig. 7, though the build-up has been represented as a decay for convenience. It will be seen that the curves are very different from those of the 6-element filter with $m = 0.884$ (see Fig. 5) ; there is a very slow commencement of the change, over 1 ms elapsing before any measurable change takes place at all. This is allowed for by making G larger. The best value for the simpler equation (22) is 0.75 and for the more complex equation (23) it is 1.0. The build-up time at the rate at the half-amplitude point is about 8.5 ms, which is somewhat less than that for the " ideal " filter. The calculated curves agree with the measured curves fairly well on this point.

It should be noted that for values of m which are not very different from 1.0 (say, for $m > 0.7$), the equations (22) and (23) are almost indistinguishable from the more complicated equations (20) and (21) given for derived sections. Since in very narrow-band filters, m values less than 0.7 are rare,

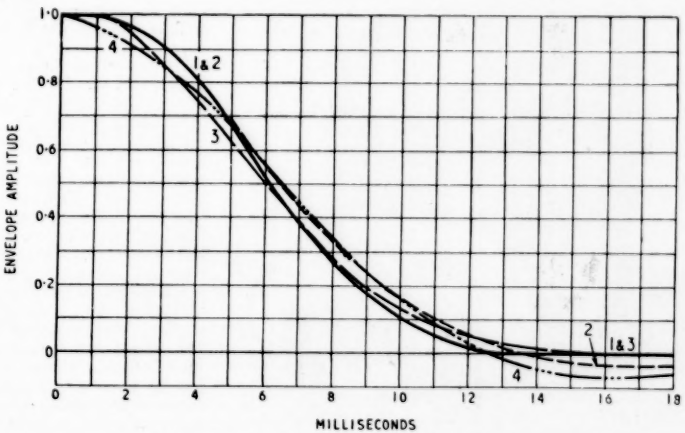


Fig. 7. *Decay curves for* $n = 0.037$, $m = 1$, $\omega_0 = 16.03 \times 10^3$. *Curve 1, measured " make " ; curve 2, measured " break " ; curve 3, calculated from* $\epsilon^{-0.203t}(1 + 0.75 \sin 0.305t)$; *curve 4, calculated from* $\epsilon^{-0.305t}(1 + \epsilon^{0.153t} \sin 0.264t)$.

the constant-k formulae can safely be used as general forms for very narrow bands. This conclusion also supports the use of the formulae for other circuit configurations. The value of G is, of course, dependent on the m value, and is the main factor distinguishing the responses of the various filter types.

4. Conclusions

A summary has been given of two methods of approach to the determination of the transient response of band-pass filters, and it has been shown that the method which considers the filter in the abstract, as an idealised transmission characteristic, gives results which are very useful for multi-section filters. For single-section filters, however, the results are not sufficiently applicable, and a special method of determination of the build-up and decay envelopes at mid-band frequency is given; the transient component of the envelope is of the form

$$\epsilon^{-A_1 t}\left[1 + G\,\epsilon^{A_2 t} \cdot \sin A_3 t\right]$$

where A_1, A_2 and A_3 are functions of the bandwidth, and m, the derivation parameter; and G is a function of m.

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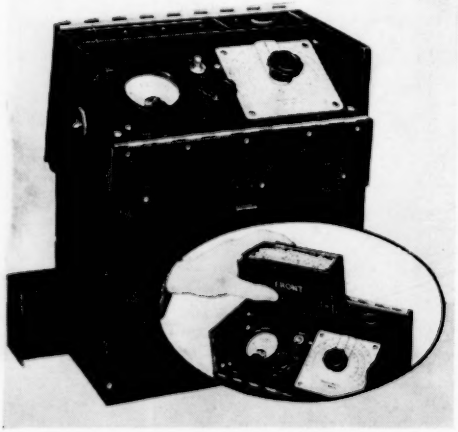
Physical Society's Exhibition

TWO errors, unfortunately, occurred in the review of this exhibition in the February issue. The butterfly-oscillator referred to on p. 59 and illustrated on p. 60 was attributed to the General Electric Co., whereas in fact it was exhibited by the British Thomson-Houston Co.



Marconi Instruments moisture-in-grain meter.

The second error occurred on p. 61; the caption to the illustration of a moisture-in-grain meter attributes it to Marconi Instruments, whereas it is actually a Mullard product. The model shown by Marconi Instruments is of quite different appearance as the accompanying illustration of their battery-driven equipment shows.

Book Review

Currents in Aerials and High Frequency Networks

By F. B. PIDDUCK. Pp. 97 with 29 Figs. Oxford University Press, Amen House, Warwick Square, London, E.C.4. 8*s*. 6*d*.

This small book is an account of investigations based on a paper published by H. C. Pocklington in 1897 and on fundamental formulae published in 1931 by F. H. Murray, an American mathematician. It is stated that the theory of transmission lines is established rigorously as that of a system of two aerials, avoiding errors [this presumably is the author's name for approximations] prevalent in elementary presentations, and considering the effect of radiation. The book is of an ultra-mathematical character, many pages being full of most forbidding looking formulae, some of which occupy a dozen lines or more. These are doubtless essential to the rigorous accuracy aimed at. The opening pages are devoted to the fundamental formulae involved, including on p. 3 the interesting subject of the ratio of the volumes occupied by electrons and atoms, but this I must confess I could not follow. The mathematical rigour does not extend to the text, for on p. 2 we are told that "it is convenient to suppose that since every electron E came originally from an atom, it from came some atom near to it." The last twelve pages are occupied by tables of values of special functions that occur in the text.

This book will doubtless appeal to those who dislike approximations in such problems and prefer rigorous mathematical accuracy at all costs.

G. W. O. H.

CORRESPONDENCE

Letters of technical interest are always welcome. In publishing such communications the Editors do not necessarily endorse any technical or general statements which they may contain.

Valve Equivalent Circuit

To the Editor, "Wireless Engineer."

SIR,—Your publication in *Wireless Engineer* of September, 1945, "Valve Equivalent Circuit Conventions and Negative Feedback" has induced me to explain my viewpoint of the problem of valve equivalent circuit conventions. The discussion on this subject arises only from inconsistency and insecurity in defining, and in the proper choice of, the signs in the circuits.

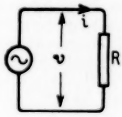


Fig. 1.

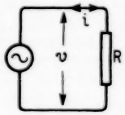


Fig. 2.

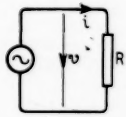


Fig. 3.

The following axioms and definitions seem to me to be basic :—

(1) The physical discussion of equations for electrical circuits is possible and correct only when the significance of the signs is defined.

(2) Alternating current and voltage change their direction periodically. If we want to write mathematical relations, we must define which direction is called positive.

(*a*) The arrow for a current is such that a positive value for this current means a current flowing in the direction of the arrow (the electrons flow in the opposite direction).

(*b*) A voltage is positive in the direction of the electric field from the positive to the negative electrode. For a positive voltage, the arrow indicates the direction of the electric field.

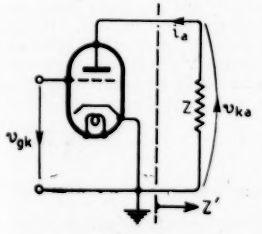


Fig. 4.

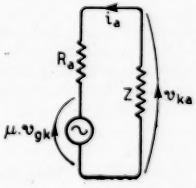


Fig. 5.

NOTE.—It is not reasonable to denote in a circuit a current with a simple arrow, and a voltage with a double one (Fig. 1). Since both quantities change their direction periodically, it would be logical to give to the current a double arrow similar to that of the voltage (Fig. 2). A clear discussion of the equations of the circuit is only possible when the indications are as in Fig. 3.

(3) The arrows for current and voltage are generally referred to each other in such a way that Ohm's law for an impedance (motor, not a generator) can be written in the positive form :

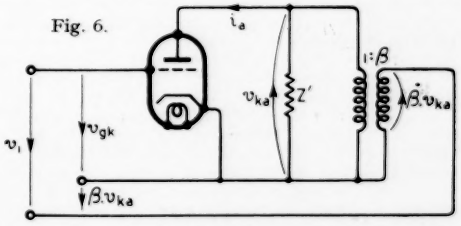
$$+v = +i \,.\, R$$

Now when the directions of positive instantaneous values for current and voltage are thus defined, the question of a positive or negative sign for the amplification factor, for the sign of the E.M.F. of the equivalent circuit, or for the direction of valve vectors, does not arise. In the case of a valve, the choice of signs is fixed and in such a way that the physical action can be seen immediately.

Let v_{gk}, i_a, v_{ka} be the instantaneous values of the alternating quantities of the valve, double suffixes giving at the same time the positive direction of the voltages. In Fig. 4, the circuit for a simple triode valve is shown, in which the direct currents and voltages are omitted for simplicity's sake. When v_{gk} is positive, i_a flows in the direction of the arrow, hence is positive. i_a produces a voltage v_{ka} across Z also in the direction of the arrow, hence v_{ka} is positive. As a physical reality,

$$i_a = \frac{\mu \,.\, v_{gk}}{R_a + Z} \qquad .. \quad .. \quad .. \quad .. \quad (1)$$

Fig. 6.



Voltage, current and their signs in the equivalent circuit have to fulfil the same equation (1). If we do not change part "Z'" in Fig. 4, we must obtain Fig. 5 with the direction indicated for the equivalent E.M.F. $\mu \,.\, v_{gk}$ of the generator. For the voltage, v_{ka} and the amplification $M = v_{ka}/v_{gk}$ we obtain

$$v_{ka} = Z' \,.\, i_a = \frac{\mu \,.\, Z' \,.\, v_{gk}}{R_a + Z'}; \quad M = \frac{v_{ka}}{v_{gk}} = \frac{\mu \,.\, Z'}{R_a + Z'}$$

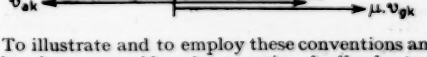
If all voltages are referred to the common cathode point, we have negative signs :

$$v_{ak} = -\, v_{ka} = \frac{\mu \,.\, Z' \,.\, v_{gk}}{R_a + Z'};$$

$$M' = \frac{v_{ak}}{v_{gk}} = -\,\frac{\mu \,.\, Z'}{R_a + Z'} = -\,M$$

These equations show that the valve acts as a phase inverter with a phase-angle of 180°.

From the above formulae the directions of the different valve vectors can be derived :



To illustrate and to employ these conventions and rules, let us consider the negative feedback stage shown in Fig. 6.

The application of equation (1) gives :

$$v_{ka} = \frac{\mu \cdot Z' \cdot v_{gk}}{R_a + Z'} \quad .. \quad .. \quad .. \quad .. \quad (2)$$

From Fig. 6 :

$$v_1 = v_{gk} + \beta \cdot v_{ka} \quad \therefore \quad v_{gk} = v_1 - \beta v_{ka} \quad .. \quad (3)$$

(3) in (2) :

$$v_{ka} = \frac{\mu \cdot Z'}{R_a + Z'} (v_1 - \beta \cdot v_{ka}) \quad .. \quad .. \quad (4)$$

From (4) :

$$v_{ka} \left(1 + \frac{\beta\mu \cdot Z'}{R_a + Z'}\right) = \frac{\mu \cdot Z'}{R_a + Z'} v_1 \quad .. \quad (5)$$

Amplification :

$$M_1 = \frac{v_{ka}}{v_1} = \frac{\dfrac{\mu \cdot Z'}{R_a + Z'}}{1 + \dfrac{\beta\mu Z'}{R_a + Z'}} = \frac{\mu Z'}{R_a + Z' (1 + \beta\mu)}$$

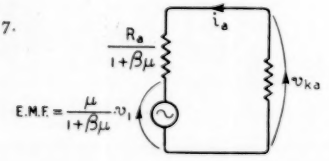
$$= \frac{\dfrac{\mu Z'}{1 + \beta\mu}}{\dfrac{R_a}{1 + \beta\mu} + Z'} \quad .. \quad .. \quad .. \quad (6)$$

Current :

$$i_a = \frac{v_{ka}}{Z'} = \frac{\dfrac{\mu \cdot v_1}{1 + \beta\mu}}{\dfrac{R_a}{1 + \beta\mu} + Z'} \quad .. \quad .. \quad (7)$$

Equation (7) gives the following values for the equivalent circuit :

Fig. 7.



HANS BIEFER.

Baden, Switzerland.

An Interesting Electrodynamical Problem

To the Editor, "Wireless Engineer."

SIR,—I have found it very interesting to compare the evaluation of the force acting on the lower limb BC of a rectangular circuit, given in your Editorial, October issue, with a similar treatment of the problem given in 1938 by O'Rahilly in his book "Electro Magnetics," Chap. IV. If I understand O'Rahilly correctly, the forces given for the action of the remaining sides of the rectangle on BC are the same, whether we choose Biot-Savart's Law or Ampère's formula ; the distinction appears when the force exerted by BC on the remaining sides is computed. Biot-Savart's Law does not give the force acting on BC equal and opposite to the force exerted by BC. If this is true it surely constitutes a very grave defect in the Biot-Savart Law.

S. RODDA.

New Barnet, Herts.

[Our correspondent is quite correct when he says that the forces given for the action of the remaining sides of the rectangle on BC are the same, whether we choose the Biot or the Ampère formula ; they both agree with the measured force. He is not so correct, however, when he says that the Biot law does not give the force acting on BC equal and opposite to the force exerted by BC. They are exactly equal and opposite and this result is given by both formulae, *if properly applied*. It is impossible to make any measurement which is not equally in agreement with the results calculated by either formula. The Ampère formula assumes action at a distance between any two current elements producing a force in the line joining the elements, but the only force on the element that can be measured is the resultant force, and that is always found to be at right-angles to the element as the Ampère formula gives when integrated around the whole circuit. The Biot formula is based on an entirely different point of view. Biot first found that the magnetic field near a long wire was inversely proportional to the distance from the wire. After Laplace had pointed out that this was explicable by assuming the field due to each element of conductor to be inversely proportional to the square of the distance and some function of the angle, Biot made further experiments and found that the required function was the sine of the angle between the element and the line joining it to the point considered. Biot measured the field strength by timing the oscillations of a magnetic needle and his formula is primarily for the determination of the field strength at any point due to any system of conductors. One can then use it to calculate the force on a current element at the point. We find the force on BC by finding the field strength at various points along it due to the whole circuit BADC. We cannot expect to obtain an equal and opposite force on any part or on the whole of the circuit by considering the magnetic field due only to BC. Just as the force on BC depends on the field in which it is situated, so the equal and opposite force on the rest of the circuit depends on the field in which it is situated and not merely on that portion of the field produced by the current in BC. When the field at all points of BADC has been found by the Biot formula, the forces on the three sides can be calculated and, since the forces on AB and CD are equal and opposite, the resultant force on the three sides is the upward force on AD. If the short side BC is fixed and the system BADC free to move, this is the force with which it will be moved upwards and this force can only be exerted by BC. It is, of course, equal and opposite to the force acting on BC.

G. W. O. H.]

Resistance Networks

To the Editor, "Wireless Engineer."

SIR,—In reading the article "Resistance Networks," by Messrs. Colchester and Gough, in *Wireless Engineer* for May, 1940, which I have only recently received, I was reminded of the fact that I do not recollect having seen in print the following simple memory-rule for the minimum insertion loss in parallel mixers with impedance matching in all directions (Fig. 10 on p. 212 of the above-mentioned article).

The rule states that with n mixers the maximum output current is one nth of the input current.

This must be so, because the output current from any particular mixer must divide equally between

the n—1 other mixers and the output circuit, all of which present the same impedance, viewed from their common junction point.

JAMES STEFFENSEN,
Technical Department,
State Broadcasting Service.
Copenhagen, Denmark.

Pulse Modulation

To the Editor, "Wireless Engineer."

SIR,—We agree with Mr. Cooke on the desirability of the early stabilization of nomenclature in the realm of pulse modulation. Without claiming originality for the following terms, many of which have gained circulation among those "skilled in the art," we would suggest these definitions and abbreviations as appearing reasonably explicit and convenient:

Term	*Abbreviation*	*Description*
Pulse Amplitude Modulation.	PAM	Amplitude only is varied by the modulation.
Pulse Length Modulation.	PLM	Total duration varied by modulation.
(*a*) Symmetrical	PLMs	Times of start and stop varied equally in opposite senses about the mean time.
(*b*) Asymmetrical.	PLM1	Time of start, only, varied.
	PLM2	Time of stop, only, varied.
Pulse Phase Modulation.	PPM	Mean time of pulse varied.
Pulse Frequency Modulation.	PFM	Pulse chain (train) frequency varied according to the modulation.
Time Allocation Multiplex.	TAM	Multichannel system in which each channel occupies the whole of the available frequency band for small consecutive intervals in rotation.
Frequency Allocation Multiplex.	FAM	Multichannel system in which each channel continuously occupies a separate fixed part of the available frequency band.

It is tacitly assumed in these descriptions that the leading and trailing edges of the pulses are substantially instantaneous and that the pulses are rectangular in form. We feel, however, that no difficulty will arise in any given case, even for such pulse shapes as the probability-function type, in deciding the appropriate description of the modulation.

We had in mind the suppression or insertion of pulses in the train when we mentioned Pulse Number Modulation, and we agree with Mr. Cooke's remarks regarding the limitations of this system.

F. F. ROBERTS.
J. C. SIMMONDS.
London, N.W.2.

Rationalization of Publications

To the Editor, "Wireless Engineer."

SIR,—I wish to thank Mr. R. E. Burgess for drawing my attention to the fact that a result which I included in a paper published in October 1941 had already been published by Mr. F. M. Colebrook in 1930; but I also wish to use this as an example of the whole problem of rationalizing scientific publications.

In an ideal world, each piece of information would be published once in an "original paper," in such a way as to be accessible to everyone interested, and thereafter would be incorporated either completely or by reference in textbooks and handbooks. But not every paper published at the present time is literally original. In a limited and detailed piece of work, I say quite frankly that if it appears novel to both author and editor, it will probably be new to a high proportion of readers: in the present example, this seems to be borne out by the fact that only after the lapse of over four years has the duplication of publication been noticed. Nevertheless, duplicate publication represents a waste of the author's time and the publisher's paper; in order to minimize waste, we need two and possibly three types of publication: (1) The full-length book on a limited branch of science or engineering—e.g. the "International Series in Physics," published by McGraw-Hill Book Co. (2) The encyclopaedia, which gives a brief outline of each topic of an extensive field, with references to more detailed work—e.g. the "Radio Handbooks" of American origin. (3) Possibly also the interim summary, to bridge the gap between original papers and the two types of book just mentioned which take a fair time to compile—e.g. the Physical Society's "Reports on Progress in Physics."

Am I being unduly pessimistic in suggesting that in the field of radio and electronics, none of these types of publication is available as a British book by a British author? Alternatively, since the better type of book is international in scope (a glance through the index of a book published in U.S.A. in 1933 showed references to Appleton, Moullin and Watson-Watt) should we welcome the extension of rationalization of publication by a further stage, and conclude that so long as the Americans do these things adequately there is no need for British engineers to give up their time to writing books? A definite policy is urgently needed now, because there is an accumulation of the greater part of six years' work which is presumably now being released for general publication.

D. A. BELL.
London, N.21.

Books Received

Inside the Vacuum Tube

By John F. Rider. Pp. 407 + xiii. John F. Rider Publisher Inc., 404, Fourth Avenue, New York, 16, N.Y. Price $4.50.

The Cathode-Ray Tube Handbook

By S. K. Lewer, B.Sc. Pp. 100 + xviii. Sir Isaac Pitman & Sons Ltd., Pitman House, Parker St., Kingsway, London, W.C.2. Price 6s.

WIRELESS PATENTS

A Summary of Recently Accepted Specifications

The following abstracts are prepared, with the permission of the Controller of H.M. Stationery Office, from Specifications obtainable at the Patent Office, 25, Southampton Buildings, London, W.C.2, price 1/- each.

ACOUSTICS AND AUDIO-FREQUENCY CIRCUITS AND APPARATUS

572 481.—Filter circuit, fed by rectifiers at two different points, to serve as a noise-reduction unit in sound-recording equipment.

Radio Corporation of America. Convention date (U.S.A.) 31st December, 1942.

572 500.—Magneto-strictive oscillator for generating sound or supersonic pressure-waves, e.g., for under-water echo-sounding.

Marconi's W. T. Co., Ltd. and H. J. Round. Application dates 24th November, 1938, and 16th May, 1939.

572 576.—Construction of acoustic conduits or chambers for dissipating or absorbing noises of high intensity.

V. Jacobsen. Application date 29th March, 1943.

572 688.—Automatic cut-out switch for the motor of a gramophone or radiogram.

E. W. Mortimer and The Garrard Engineering and Manufacturing Co., Ltd. Application date 5th November, 1943.

DIRECTIONAL WIRELESS

572 470.—Capacitance-balancing arrangement for a loop-and-vertical aerial, to ensure a clear-cut cardioid response on ultra-short waves.

Marconi's W. T. Co., Ltd. and F. M. Wright. Application date 7th July, 1941.

572 508.—Training equipment for simulating the conditions under which D. F. observations are taken on a mobile craft from a fixed transmitter.

Rediffusion Ltd. and P. Adorjan. Application date, 21st February, 1940.

572 570.—Training equipment for simulating the conditions under which D. F. observations are taken on a number of fixed beacon-stations.

Rediffusion Ltd., P. Adorjan and R. P. Gabriel. Application date, 11th July, 1940.

572 912.—Indicator circuit for sharply distinguishing the "dot" from the "dash" signals, say in a blind-landing beam system.

Standard Telephones and Cables Ltd., C. M. le G. Eyre and D. Hamilton. Application date, 11th February, 1944.

RECEIVING CIRCUITS AND APPARATUS

572 426.—Means for ensuring an accurately-graded sliding movement of the powdered-iron cores of ganged tuning inductances.

Zenith Radio Corp. Convention date (U.S.A.), 14th September, 1942.

572 511.—Mounting and encasing the two elements of a crystal detector, particularly for use as a mixer when receiving very short waves.

The General Electric Co., Ltd., F. H. Brittain and C. E. Ransley. Application date, 5th November, 1941.

572 955.—Portable valve-receiver in which the listener's body is coupled, as an aerial, to the tuned input circuit through the headphones, and the dry batteries serve as a counterpoise.

H. Jackson. Application date, 27th April, 1942.

TRANSMITTING CIRCUITS AND APPARATUS

572 781.—Construction of a R. F. tuning chamber provided with fixed and movable sheets or septums to vary its resonant frequency.

E. B. Moullin and Metropolitan-Vickers Electrical Co. Ltd. Application date, 30th April, 1943.

572 783.—Loud-speaker installation coupled to the circuit of a ship's radio-transmitter.

International Marine Radio Ltd. and C. G. G. Withey. Application date, 21st February, 1944.

SUBSIDIARY APPARATUS AND MATERIALS

572 163.—Remote-control device comprising a fixed ring-magnet and a rotary bar-magnet for indicating the setting of a distant condenser or potentiometer.

Standard Telephones and Cables, Ltd. and S. H. Towner. Application date 5th November, 1943.

572 208.—Remote control and indicating equipment for exploring and recording the condition of a series of electrical track or like circuits.

Standard Telephones and Cables, Ltd. (communicated by International Standard Electric Corporation). Application date 22nd October, 1943.

572 292.—Process for "spot-welding" or bonding together sheets of dielectric materials by high-frequency current.

Radio Corporation of America. Convention date (U.S.A.) 31st July, 1942.

572 324.—Hand-tool for connecting wires together, or to circuit components, by electric welding or fusion.

Standard Telephones and Cables, Ltd. and G. Gilliver. Application date 20th March, 1944.

572 329.—Diphenyl composition, particularly for use as the impregnating dielectric for "interleaved" condensers.

The British Thomson-Houston Co., Ltd. Convention date (U.S.A.) 25th February, 1942.

572 398.—Hand-tool of the pincers type for stripping electric cables.

Dorman and Smith, Ltd., and T. Atherton. Application date 30th March, 1944.

572 443.—Hand-tool for "spot-welding" sheets of dielectric materials by radio-frequency currents.

Radio Corporation of America. Convention date (U.S.A.) 31st July, 1942.